

# TAME AUTOMORPHISMS FIXING A VARIABLE OF FREE ASSOCIATIVE ALGEBRAS OF RANK THREE

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**ABSTRACT.** We study automorphisms of the free associative algebra  $K\langle x, y, z \rangle$  over a field  $K$  which fix the variable  $z$ . We describe the structure of the group of  $z$ -tame automorphisms and derive algorithms which recognize  $z$ -tame automorphisms and  $z$ -tame coordinates.

## INTRODUCTION

Let  $K$  be an arbitrary field of any characteristic and let  $K[x_1, \dots, x_n]$  and  $K\langle x_1, \dots, x_n \rangle$  be, respectively, the polynomial algebra in  $n$  variables and of the free associative algebra of rank  $n$ , freely generated by  $x_1, \dots, x_n$ . We may think of  $K\langle x_1, \dots, x_n \rangle$  as the algebra of polynomials in  $n$  noncommuting variables. The automorphism groups  $\text{Aut } K[x_1, \dots, x_n]$  and  $\text{Aut } K\langle x_1, \dots, x_n \rangle$  are well understood for  $n \leq 2$  only. The description is trivial for  $n = 1$ , when the automorphisms  $\varphi$  are defined by  $\varphi(x_1) = \alpha x_1 + \beta$ , where  $\alpha \in K^* = K \setminus \{0\}$  and  $\beta \in K$ . The classical results of Jung–van der Kulk [J, K] for  $K[x_1, x_2]$  and of Czerniakiewicz–Makar-Limanov [Cz, ML1, ML2] give that all automorphisms of  $K[x_1, x_2]$  and  $K\langle x_1, x_2 \rangle$  are tame. Writing the automorphisms of  $K[x_1, \dots, x_n]$  and  $\text{Aut } K\langle x_1, \dots, x_n \rangle$  as  $n$ -tuples of the images of the variables, and using  $x, y$  instead of  $x_1, x_2$ , this means that  $\text{Aut } K[x, y]$  and  $\text{Aut } K\langle x, y \rangle$  are generated by the affine automorphisms

$$\psi = (\alpha_{11}x + \alpha_{21}y + \beta_1, \alpha_{12}x + \alpha_{22}y + \beta_2), \quad \alpha_{ij}, \beta_j \in K,$$

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(and  $\psi_1 = (\alpha_{11}x + \alpha_{21}y, \alpha_{12}x + \alpha_{22}y)$ , the linear part of  $\psi$ , is invertible) and the triangular automorphisms

$$\rho = (\alpha_1x + p_1(y), \alpha_2y + \beta_2), \quad \alpha_1, \alpha_2 \in K^*, p_1(y) \in K[y], \beta_2 \in K.$$

It turns out that the groups  $\text{Aut } K\langle x, y \rangle$  and  $\text{Aut } K[x, y]$  are naturally isomorphic. As abstract groups they are described as the free product  $A *_C B$  of the group  $A$  of the affine automorphisms and the group  $B$  of triangular automorphisms amalgamating their intersection  $C = A \cap B$ . Every automorphism  $\varphi$  of  $K[x, y]$  and  $K\langle x, y \rangle$  can be presented as a product

$$(1) \quad \varphi = \psi_m^{\varepsilon_m} \rho_m \psi_{m-1} \cdots \rho_2 \psi_1 \rho_1^{\varepsilon_1},$$

where  $\psi_i \in A$ ,  $\rho_i \in B$  ( $\varepsilon_1$  and  $\varepsilon_m$  are equal to 0 or 1), and, if  $\varphi$  does not belong to the union of  $A$  and  $B$ , we may assume that  $\psi_i \in A \setminus B$ ,  $\rho_i \in B \setminus A$ . The freedom of the product means that if  $\varphi$  has a nontrivial presentation of this form, then it is different from the identity automorphism.

In the case of arbitrary  $n$ , the tame automorphisms are defined similarly, as compositions of affine and triangular automorphisms. One studies not only the automorphisms but also the coordinates, i.e., the automorphic images of  $x_1$ .

We shall mention few facts related with the topic of the present paper, for  $z$ -automorphisms of  $K[x, y, z]$  and  $K\langle x, y, z \rangle$ , i.e., automorphisms fixing the variable  $z$ . For more details we refer to the books by van den Essen [E], Mikhalev, Shpilrain, and Yu [MSY], and our survey article [DY1].

Nagata [N] constructed the automorphism of  $K[x, y, z]$

$$\nu = (x - 2(y^2 + xz)y - (y^2 + xz)^2z, y + (y^2 + xz)z, z)$$

which fixes  $z$ . He showed that  $\nu$  is nontame, or wild, considered as an automorphism of  $K[z][x, y]$ , and conjectured that it is wild also as an element of  $\text{Aut } K[x, y, z]$ . This was the beginning of the study of  $z$ -automorphisms.

It is relatively easy to see (and to decide algorithmically) whether an endomorphism of  $K[z][x, y]$  is an automorphism and whether this automorphism is  $z$ -tame, or tame as an automorphism of  $K[z][x, y]$ . When  $\text{char } K = 0$ , Drensky and Yu [DY2] presented a simple algorithm which decides whether a polynomial  $f(x, y, z) \in K[x, y, z]$  is a  $z$ -coordinate and whether this coordinate is  $z$ -tame. This provided many new wild automorphisms and wild coordinates of  $K[z][x, y]$ . These results in

[DY2] are based on a similar algorithm of Shpilrain and Yu [SY1] which recognizes the coordinates of  $K[x, y]$ . Shestakov and Umirbaev [SU1, SU2, SU3] established that the Nagata automorphism is wild. They also showed that every wild automorphism of  $K[z][x, y]$  is wild as an automorphism of  $K[x, y, z]$ . Umirbaev and Yu [UY] proved that the  $z$ -wild coordinates in  $K[z][x, y]$  are wild also in  $K[x, y, z]$ . In this way, all  $z$ -wild examples in [DY2] give automatically wild examples in  $K[x, y, z]$ .

Going to free algebras, the most popular candidate for a wild automorphism of  $K\langle x, y, z \rangle$  is the example of Anick  $(x + (y(xy - yz)), y, z + (zy - yz)y) \in \text{Aut } K\langle x, y, z \rangle$ , see the book by Cohn [C], p. 343. It fixes one variable and its abelianization is a tame automorphism of  $K[x, y, z]$ . Exchanging the places of  $y$  and  $z$ , we obtain the automorphism  $(x + z(xz - zy), y + (xz - zy)z, z)$  which fixes  $z$  (or a  $z$ -automorphism), and refer to it as the Anick automorphism. It is linear in  $x$  and  $y$ , considering  $z$  as a “noncommutative constant”. Drensky and Yu [DY3] showed that such  $z$ -automorphisms are  $z$ -wild if and only if a suitable invertible  $2 \times 2$  matrix with entries from  $K[z_1, z_2]$  is not a product of elementary matrices. In particular, this gives that the Anick automorphism is  $z$ -wild. When  $\text{char } K = 0$ , Umirbaev [U] described the defining relations of the group of tame automorphisms of  $K[x, y, z]$ . He showed that  $\varphi = (f, g, h) \in \text{Aut } K\langle x, y, z \rangle$  is wild if the endomorphism  $\varphi_0 = (f_0, g_0, z)$  of  $K\langle x, y, z \rangle$  is a  $z$ -wild automorphism, where  $f_0, g_0$  are the linear in  $x, y$  components of  $f, g$ , respectively. This implies that the Anick automorphism is wild. Recently Drensky and Yu [DY4, DY5] established the wildness of a big class of automorphisms and coordinates of  $K\langle x, y, z \rangle$ . Many of them cannot be handled with direct application of the methods of [DY3] and [U]. These results motivate the needs of systematic study of  $z$ -automorphisms of  $K\langle x, y, z \rangle$ . As in the case of  $z$ -automorphisms of  $K[x, y, z]$ , they are simpler than the arbitrary automorphisms of  $K\langle x, y, z \rangle$  and provide important examples and conjectures for  $\text{Aut } K\langle x, y, z \rangle$ .

In the present paper we describe the structure of the group of  $z$ -tame automorphisms of  $K\langle x, y, z \rangle$  as the free product of the groups of  $z$ -affine automorphisms and  $z$ -triangular automorphisms amalgamating the intersection. We also give algorithms which recognize  $z$ -tame automorphisms and coordinates of  $K\langle x, y, z \rangle$ . As an application, we show that all the  $z$ -automorphisms of the form  $\sigma_h = (x + zh(xz - zy), y + h(xz - zy), z)$  are  $z$ -wild when the polynomials  $h(xz - zy, z)$  are of positive

degree in  $x$ . This kind of automorphisms appear in [DY4, DY5] but the considerations there do not cover the case when  $h(xz - zy, z)$  belongs to the square of the commutator ideal of  $K\langle x, y, z \rangle$ . Besides, the polynomial  $x + zh(xz - zy, z)$  is a  $z$ -wild coordinate. Finally, we show that the  $z$ -endomorphisms of the form  $\varphi = (x + u(x, y, z), y + v(x, y, z))$ , where  $(u, v) \neq (0, 0)$  and all monomials of  $u$  and  $v$  depend on both  $x$  and  $y$ , are not automorphisms. A partial case of this result was an essential step in the proof of the theorem of Czerniakiewicz and Makar-Limanov for the tameness of  $\text{Aut } K\langle x, y \rangle$ . The paper may be considered as a continuation of our paper [DY3].

### 1. THE GROUP OF $z$ -TAME AUTOMORPHISMS

We fix the field  $K$  and consider the free associative algebra  $K\langle x, y, z \rangle$  in three variables. We call the automorphism  $\varphi$  of  $K\langle x, y, z \rangle$  a  $z$ -automorphism if  $\varphi(z) = z$ , and denote the automorphism group of the  $z$ -automorphisms by  $\text{Aut}_z\langle x, y, z \rangle$ . Since we want to emphasize that we work with  $z$ -automorphisms, we shall write  $\varphi = (f, g)$ , omitting the third coordinate  $z$ . The multiplication will be from right to left. If  $\varphi, \psi \in \text{Aut}_z K\langle x, y, z \rangle$ , then in  $\varphi\psi$  we first apply  $\psi$  and then  $\varphi$ . Hence, if  $\varphi = (f, g)$  and  $\psi = (u, v)$ , then

$$\varphi\psi = (u(f, g, z), v(f, g, z)).$$

The  $z$ -affine and  $z$ -triangular automorphisms of  $K\langle x, y, z \rangle$  are, respectively, of the form

$$\psi = (\alpha_{11}x + \alpha_{21}y + \alpha_{31}z + \beta_1, \alpha_{12}x + \alpha_{22}y + \alpha_{32}z + \beta_2),$$

$\alpha_{ij}, \beta_j \in K$ , the  $2 \times 2$  matrix  $(\alpha_{ij})_{i,j=1,2}$  being invertible,

$$\rho = (\alpha_1x + p_1(y, z), \alpha_2y + p_2(z)),$$

$\alpha_j \in K^*$ ,  $p_1 \in K\langle y, z \rangle$ ,  $p_2 \in K[z]$ . The affine and the triangular  $z$ -automorphisms generate, respectively, the subgroups  $A_z$  and  $B_z$  of  $\text{Aut}_z K\langle x, y, z \rangle$ . We denote by  $\text{TAut}_z K\langle x, y, z \rangle$  the group of  $z$ -tame automorphisms which is generated by the  $z$ -affine and  $z$ -triangular automorphisms. Of course, we may define the  $z$ -affine automorphisms as the  $z$ -automorphisms of the form  $\psi = (f, g)$ , where the polynomials  $f, g \in K\langle x, y, z \rangle$  are linear in  $x$  and  $y$ . But, as we commented in [DY3], this definition is not convenient. For example, the Anick automorphism is affine in this sense but is wild.

In the commutative case, the  $z$ -automorphisms of  $K[x, y, z]$  are simply the automorphisms of the  $K[z]$ -algebra  $K[z][x, y]$ . A result of

Wright [Wr] states that over any field  $K$  the group  $\text{TAut}_z K[x, y, z]$  has the amalgamated free product structure

$$\text{TAut}_z K[x, y, z] = A_z *_{C_z} B_z,$$

where  $A_z$  and  $B_z$  are defined as in the case of  $K\langle x, y, z \rangle$  and  $C_z = A_z \cap B_z$ . (The original statement in [Wr] holds in a more general situation. In the case of  $K[x, y, z]$  it involves affine and linear automorphisms with coefficients from  $K[z]$  but this is not essential because every invertible matrix with entries in  $K[z]$  is a product of elementary and diagonal matrices.)

Every  $z$ -tame automorphism  $\varphi$  of  $K\langle x, y, z \rangle$  can be presented as a product in the form (1) where  $\psi_i \in A_z$ ,  $\rho_i \in B_z$  ( $\varepsilon_1$  and  $\varepsilon_m$  are equal to 0 or 1), and, if  $\varphi$  does not belong to the union of  $A_z$  and  $B_z$ , we may assume that  $\psi_i \in A_z \setminus B_z$ ,  $\rho_i \in B_z \setminus A_z$ . Fixing the linear nontriangular  $z$ -automorphism  $\tau = (y, x)$ , we can present  $\varphi$  in the canonical form

$$(2) \quad \varphi = \rho_n \tau \cdots \tau \rho_1 \tau \rho_0,$$

where  $\rho_0, \rho_1, \dots, \rho_n \in B_z$  and only  $\rho_0$  and  $\rho_n$  are allowed to belong to  $A_z$ , see for example p. 350 in [C]. Let

$$\rho_i = (\alpha_i x + p_i(y, z), \beta_i y + r_i(z)), \quad \alpha_i, \beta_i \in K^*, p_i \in K\langle y, z \rangle, r_i \in K[z].$$

Using the equalities for compositions of automorphisms

$$(\alpha x + p(y, z), \beta y + r(z)) = (x + \alpha^{-1}(p(y, z) - p(0, z)), y)(\alpha x + p(0, z), \beta y + r(z)),$$

$$(\alpha x + p(z), \beta y + r(z))\tau = (\beta y + r(z), \alpha x + p(z)) = \tau(\beta x + r(z), \alpha y + p(z)),$$

$p(z), r(z) \in K[z]$ , we can do further simplifications in (2), assuming that  $\rho_1, \dots, \rho_{n-1}$  are not affine and, together with  $\rho_n$ , are of the form  $\rho_i = (x + p_i(y, z), y)$  with  $p_i(0, z) = 0$  for all  $i = 1, \dots, n$ . We also assume that  $\rho_0 = (\alpha_0 x + p_0(y, z), \beta_0 y + r_0(z))$ . The condition that  $\rho_1, \dots, \rho_{n-1}$  are not affine means that  $\deg_y p_i(y, z) \geq 1$  and if  $\deg_y p_i(y, z) = 1$ , then  $\deg_z p_i(y, z) \geq 1$ ,  $i = 1, \dots, n-1$ .

The following result shows that the structure of the group of  $z$ -tame automorphisms of  $K\langle x, y, z \rangle$  is similar to the structure of the group of  $z$ -tame automorphisms of  $K[x, y, z]$ .

**Theorem 1.1.** *Over an arbitrary field  $K$ , the group  $\text{TAut}_z K\langle x, y, z \rangle$  of  $z$ -tame automorphisms of  $K\langle x, y, z \rangle$  is isomorphic to the free product  $A_z *_{C_z} B_z$  of the group  $A_z$  of the  $z$ -affine automorphisms and the group  $B_z$  of  $z$ -triangular automorphisms amalgamating their intersection  $C_z = A_z \cap B_z$ .*

*Proof.* We define a bidegree of  $K\langle x, y, z \rangle$  assuming that the monomial  $w$  is of bidegree  $\text{bideg } w = (d, e)$  if  $\deg_x w + \deg_y w = d$  and  $\deg_z w = e$ . We order the bidegrees  $(d, e)$  lexicographically, i.e.,  $(d_1, e_1) > (d_2, e_2)$  means that either  $d_1 > d_2$  or  $d_1 = d_2$  and  $e_1 > e_2$ . We denote by  $\bar{p}$  the leading bihomogeneous component of the nonzero polynomial  $p(x, y, z)$ . Let  $\varphi = (f, g)$  be in the form (2), with all the restrictions fixed above, and let  $q_i(y, z)$  be the leading component of  $p_i(y, z)$ . Direct computations give that, if  $\rho_n$  is not linear and  $p_0(y, z) \neq \gamma_0 y + p'_0(z)$  in  $\rho_0 = (\alpha_0 x + p_0(y, z), \beta_0 y + r_0(z))$ , then

$$(3) \quad \begin{aligned} \bar{f} &= q_0(q_1(\dots q_{n-1}(q_n(y, z), z) \dots, z), z), \\ \bar{g} &= \beta_0 q_1(\dots q_{n-1}(q_n(y, z), z) \dots, z), \end{aligned}$$

and  $\text{bideg } \bar{f} > (1, 0)$ . Hence  $\varphi$  is not the identity automorphism. Similar considerations work when at least one of the automorphisms  $\rho_0$  and  $\rho_n$  is affine. For example, if  $\rho_0 = (\alpha_0 + \gamma_0 y + p'_0(z), \beta_0 y + r_0(z))$ ,  $\gamma_0 \in K^*$ , and  $\text{bideg } p_n(y, z) > (1, 0)$ , then

$$\begin{aligned} \bar{f} &= \gamma_0 q_1(\dots q_{n-1}(q_n(y, z), z) \dots, z), \\ \bar{g} &= \beta_0 q_1(\dots q_{n-1}(q_n(y, z), z) \dots, z). \end{aligned}$$

If  $\text{bideg } p_0(y, z) > (1, 0)$  and  $\rho_n = (x + \gamma_n y, y)$ ,  $\gamma_n \in K^*$ , then

$$\begin{aligned} \bar{f} &= q_0(q_1(\dots q_{n-1}(q_n(x + \gamma_n y, z), z) \dots, z), z), \\ \bar{g} &= \beta_0 q_1(\dots q_{n-1}(q_n(x + \gamma_n y, z), z) \dots, z). \end{aligned}$$

In all the cases,  $\varphi$  is not the identity automorphism. Hence, if  $\varphi$  has a nontrivial presentation in the form (2), then it is different from the identity automorphism, and we conclude that  $\text{TAut}_z K\langle x, y, z \rangle$  is a free product with amalgamation of the groups  $A_z$  and  $B_z$ .  $\square$

Following our paper [DY3] we identify the group of  $z$ -automorphisms which are linear in  $x$  and  $y$  with the group  $GL_2(K[z_1, z_2])$ . Let  $f \in K\langle x, y, z \rangle$  be linear in  $x, y$ . Then  $f$  has the form

$$f = \sum \alpha_{ij} z^i x z^j + \sum \beta_{ij} z^i y z^j, \quad \alpha_{ij}, \beta_{ij} \in K.$$

The  $z$ -derivatives  $f_x$  and  $f_y$  are defined by

$$f_x = \sum \alpha_{ij} z_1^i z_2^j, \quad f_y = \sum \beta_{ij} z_1^i z_2^j.$$

Here  $f_x$  and  $f_y$  are in  $K[z_1, z_2]$  and are polynomials in two commuting variables. The  $z$ -Jacobian matrix of the linear  $z$ -endomorphism  $\varphi =$

$(f, g)$  of  $K\langle x, y, z \rangle$  is defined as

$$J_z(\varphi) = \begin{pmatrix} f_x & g_x \\ f_y & g_y \end{pmatrix}.$$

By [DY3] the mapping  $\varphi \rightarrow J_z(\varphi)$  is an isomorphism of the group of the  $z$ -automorphisms which are linear in  $x, y$  and  $GL_2(K[z_1, z_2])$ . Also, such an automorphism is  $z$ -tame if and only if its  $z$ -Jacobian matrix belongs to  $GE_2(K[z_1, z_2])$ . (By the further development of this result by Umirbaev [U], the  $z$ -wild automorphisms of the considered type are wild also as automorphisms of  $K\langle x, y, z \rangle$ .)

**Corollary 1.2.** *The group  $\text{TAut}_z K\langle x, y, z \rangle$  is isomorphic to the free product with amalgamation  $GE_2(K[z_1, z_2]) *_{C_1} B_z$ , where  $GE_2(K[z_1, z_2])$  is identified as above with the group of  $z$ -tame automorphisms which are linear in  $x$  and  $y$ , and  $C_1 = GE_2(K[z_1, z_2]) \cap B_z$ .*

*Proof.* Everything follows from the observations that: (i) in the form (2),  $\rho_j \tau \cdots \tau \rho_i \in GE_2(K[z_1, z_2])$  if and only if all  $\rho_j, \dots, \rho_i$  belong to  $GE_2(K[z_1, z_2])$ ; (ii)  $\rho_j \tau \cdots \tau \rho_i \in C_1$  if and only if  $i = j$  and  $\rho_i \in GE_2[z_1, z_2]$ ; (iii)  $\tau \in GE_2(K[z_1, z_2])$ .  $\square$

## 2. RECOGNIZING $z$ -TAME AUTOMORPHISMS AND COORDINATES

Now we use Theorem 1.1 to present algorithms which recognize  $z$ -tame automorphisms and coordinates of  $K\langle x, y, z \rangle$ . Of course, in all algorithms we assume that the field  $K$  is constructive. We start with an algorithm which determines whether a  $z$ -endomorphism of  $K\langle x, y, z \rangle$  is a  $z$ -tame automorphism. The main idea is similar to that of the well known algorithm which decides whether an endomorphism of  $K[x, y]$  is an automorphism, see Theorem 6.8.5 in [C], but the realization is more sophisticated. In order to simplify the considerations, we shall use the trick introduced by Formanek [F] in his construction of central polynomials of matrices.

Let  $H_n$  be the subspace of  $K\langle x, y, z \rangle$  consisting of all polynomials which are homogeneous of degree  $n$  with respect to  $x$  and  $y$ . We define an action of  $K[t_0, t_1, \dots, t_n]$  on  $H_n$  in the following way. If

$$w = z^{a_0} u_1 z^{a_1} u_2 \cdots z^{a_{n-1}} u_n z^{a_n},$$

where  $u_i = x$  or  $u_i = y$ ,  $i = 1, \dots, n$ , then

$$t_0^{b_0} t_1^{b_1} \cdots t_n^{b_n} * w = z^{a_0+b_0} u_1 z^{a_1+b_1} u_2 \cdots z^{a_{n-1}+b_{n-1}} u_n z^{a_n+b_n},$$

and then extend this action by linearity. Clearly,  $H_n$  is a free  $K[t_0, t_1, \dots, t_n]$ -module with basis consisting of the  $2^n$  monomials  $u_1 \cdots u_n$ , where  $u_i = x$  or  $u_i = y$ . The proof of the following lemma is obtained by easy direct computation.

**Lemma 2.1.** *Let  $\beta \in K^*$ ,*

$$(4) \quad v(x, y, z) = \sum \theta_i(t_0, t_1, \dots, t_k) * u_{i_1} \cdots u_{i_k} \in H_k,$$

$$(5) \quad q(y, z) = \omega(t_0, t_1, \dots, t_d) * y^d \in H_d,$$

where  $\theta_i \in K[t_0, t_1, \dots, t_k]$ ,  $\omega \in K[t_0, t_1, \dots, t_d]$ ,  $u_{i_j} = x$  or  $u_{i_j} = y$ .  
Then

$$(6) \quad \begin{aligned} u(x, y, z) &= q(v(x, y, z)/\beta, z) = \omega(t_0, t_d, t_{2d}, \dots, t_{kd})/\beta^d \\ &\quad (\sum \theta_i(t_0, t_1, \dots, t_k) * u_{i_1} \cdots u_{i_k}) \\ &\quad (\sum \theta_i(t_k, t_{k+1}, \dots, t_{2k}) * u_{i_1} \cdots u_{i_k}) \cdots \\ &\quad (\sum \theta_i(t_{k(d-1)}, t_{k(d-1)+1}, \dots, t_{kd}) * u_{i_1} \cdots u_{i_k}). \end{aligned}$$

**Algorithm 2.2.** Let  $\varphi = (f, g)$  be a  $z$ -endomorphism of  $K\langle x, y, z \rangle$ . We make use of the bidegree defined in the proof of Theorem 1.1.

*Step 0.* If some of the polynomials  $f, g$  depends on  $z$  only, then  $\varphi$  is not an automorphism.

*Step 1.* Let  $u, v$  be the homogeneous components of highest bidegree of  $f, g$ , respectively. If both  $u, v$  are of bidegree  $(1, 0)$ , i.e., linear, then we check whether they are linearly independent. If yes, then  $\varphi$  is a product of a linear automorphism (from  $GL_2(K)$ ) and a translation  $(x + p(z), y + r(z))$ . If  $u, v$  are linearly dependent, then  $\varphi$  is not an automorphism.

*Step 2.* Let  $\text{bideg } u > (1, 0)$  and  $\text{bideg } u \geq \text{bideg } v$ . Hence  $u \in H_l$ ,  $v \in H_k$  for some  $k$  and  $l$ . Taking into account (3), we have to check whether  $l = kd$  for a positive integer  $d$  and to decide whether  $u = q(v/\beta, z)$  for some  $\beta \in K^*$  and some  $q(y, z) \in H_d$ . In the notation of Lemma 2.1, we know  $u$  in (6) and  $v$  in (4) up to the multiplicative constant  $\beta$ . Hence, up to  $\beta$ , we know the polynomials  $\theta_i(t_0, t_1, \dots, t_n)$  in the presentation of  $v$ . We compare some of the nonzero polynomial coefficients of  $u = \sum \lambda_j(t_0, \dots, t_{kd}) u_{j_1} \cdots u_{i_{kd}}$  with the corresponding coefficient of  $q(v/\beta, z)$ . Lemma 2.1 allows to find explicitly, up to the value of  $\beta^d$ , the polynomial  $\omega(t_0, t_1, \dots, \omega_d)$  in (5) using the usual

division of polynomials. If  $l = kd$  and  $u = q(v/\beta, z)$ , then we replace  $\varphi = (f, g)$  with  $\varphi_1 = (f - q(g/\beta, z), g)$ . Then we apply Step 0 to  $\varphi_1$ . If  $u$  cannot be presented in the desired form, then  $\varphi$  is not an automorphism.

*Step 3.* If  $\text{bideg } v > (1, 0)$  and  $\text{bideg } u < \text{bideg } v$ , we have similar considerations, as in Step 2, replacing  $\varphi = (f, g)$  with  $\varphi_1 = (f, g - q(f/\alpha, z))$  for suitable  $q(y, z)$ . Then we apply Step 0 to  $\varphi_1$ . If  $v$  cannot be presented in this form, then  $\varphi$  is not an automorphism.

**Corollary 2.3.** *Let  $h(t, z) \in K\langle t, z \rangle$  and let  $\deg_u h(u, z) > 0$ . Then*

$$\sigma_h = (x + zh(xz - zy, z), y + h(xz - zy, z)z, z)$$

*is a  $z$ -wild automorphism of  $K\langle x, y, z \rangle$ .*

*Proof.* It is easy to see that  $\sigma_h$  is a  $z$ -automorphism of  $K\langle x, y, z \rangle$  with inverse  $\sigma_{-h}$ . We apply Algorithm 2.2. Let  $w$  be the homogeneous component of highest bidegree of  $h(xz - zy, z)$ . Clearly,  $w$  has the form  $w = \bar{h}(xz - zy, z) = q(xz - zy, z)$  for some bihomogeneous polynomial  $q(t, z) \in K\langle t, z \rangle$ . The leading components of the coordinates of  $\sigma_h$  are  $zq(xz - zy, z)$  and  $q(xz - zy, z)z$ , and are of the same bidegree. If  $\sigma_h$  is a  $z$ -tame automorphism, then we can reduce the bidegree using a linear transformation, which is impossible because  $zq(xz - zy, z)$  and  $q(xz - zy, z)z$  are linearly independent.  $\square$

The algorithm in Theorem 6.8.5 in [C] which recognizes the automorphisms of  $K[x, y]$  can be easily modified to recognize the coordinates of  $K[x, y]$ . Such an algorithm is explicitly stated in [SY3], where Shpilrain and Yu established an algorithm which gives a canonical form, up to automorphic equivalence, of a class of polynomials in  $K[x, y]$ . (The automorphic equivalence problem for  $K[x, y]$  asks how to decide whether, for two given polynomials  $p, q \in K[x, y]$ , there exists an automorphism  $\varphi$  such that  $q = \varphi(p)$ . It was solved over  $\mathbb{C}$  by Wightwick [Wi] and, over an arbitrary algebraically closed constructive field  $K$ , by Makar-Limanov, Shpilrain, and Yu [MLSY].) When  $\text{char } K = 0$ , Shpilrain and Yu [SY1] gave a very simple algorithm which decides whether a polynomial  $f(x, y) \in K[x, y]$  is a coordinate. Their approach is based on an idea of Wright [Wr] and the Euclidean division algorithm applied for the partial derivatives of a polynomial in  $K[x, y]$ . Using the isomorphism of  $\text{Aut } K[x, y]$  and  $\text{Aut } K\langle x, y \rangle$  and reducing the considerations to the case of  $K[x, y]$ , Shpilrain and Yu [SY2] found the first algorithm which recognizes the coordinates of  $K\langle x, y \rangle$ . Now we want to modify

Algorithm 2.2 to decide whether a polynomial  $f(x, y, z)$  is a  $z$ -tame coordinate of  $K\langle x, y, z \rangle$ .

Note, that if  $\varphi = (f, g)$  and  $\varphi' = (f, g')$  are two  $z$ -automorphisms of  $K\langle x, y, z \rangle$  with the same first coordinate, then  $\varphi^{-1}\varphi'$  fixes  $x$ . Hence  $\varphi^{-1}\varphi' = (x, g'')$  and, obligatorily,  $g'' = \beta y + r(x, z)$ . In this way, if we know one  $z$ -coordinate mate  $g$  of  $f$ , then we are able to find all other  $z$ -coordinate mates. These arguments and Corollary 2.3 give immediately:

**Corollary 2.4.** *Let  $h(t, z) \in K\langle t, z \rangle$  and let  $\deg_u h(u, z) > 0$ . Then  $f(x, y, z) = x + zh(xz - zy, z)$  is a  $z$ -wild coordinate of  $K\langle x, y, z \rangle$ .*

**Theorem 2.5.** *There is an algorithm which decides whether a polynomial  $f(x, y, z) \in K\langle x, y, z \rangle$  is a  $z$ -tame coordinate.*

*Proof.* We start with the analysis of the behavior of the first coordinate  $f$  of  $\varphi$  in (2). Let  $h$  be the first coordinate of  $\psi = \rho_{n-1}\tau \cdots \tau\rho_1\tau\rho_0$  and let, as in (2),  $\rho_n = (x + p_n(y, z), y)$  and  $p_n(0, z) = 0$ . Then

$$(7) \quad f(x, y, z) = \rho_n\tau(h(x, y, z)) = h(y, x + p_n(y, z), z).$$

In order to make the inductive step, we have to recover the polynomials  $h(x, y, z)$  and  $p_n(y, z)$  or, at least their leading components with respect to a suitable grading.

For a pair of positive integers  $(a, b)$ , we define the  $(a, b)$ -bidegree of a monomial  $w \in K\langle x, y, z \rangle$  by

$$\text{bideg}_{(a,b)} w = (a\deg_x w + b\deg_y w, \deg_z w)$$

and order the bidegrees in the lexicographic order, as in Algorithm 2.2. For a nonzero polynomial  $f \in K\langle x, y, z \rangle$  we denote by  $|f|_{(a,b)}$  the homogeneous component of maximal  $(a, b)$ -bidegree. We write  $\varphi = (f, g) \in \text{TAut}_z K\langle x, y, z \rangle$  in the form (2). Let us assume again that  $\text{bideg } p_i(y) > (1, 0)$  for all  $i = 0, 1, \dots, n$ , and let  $h$  be the first coordinate of  $\psi = \rho_{n-1}\tau \cdots \tau\rho_1\tau\rho_0$ . Then the highest bihomogeneous component of  $h$  is

$$\bar{h}(y, z) = q_0(q_1(\dots(q_{n-1}(y, z), z)\dots), z).$$

The homogeneous component of maximal  $(d_n, 1)$ -bidegree of  $x + p_n(y, z)$  is  $|x + q_n(y, z)|_{(d_n, 1)} = x + \xi_n y^{d_n}$  if  $\deg_z q_n(y, z) = 0$  and  $|x + q_n(y, z)|_{(d_n, 1)} = q_n(y, z)$  if  $\deg_z q_n(y, z) > 0$ . Direct calculations give

$$|f|_{(d_n, 1)} = |\rho_n\tau(\bar{h})|_{(d_n, 1)} = |\bar{h}(x + q_n(y, z))|_{(d_n, 1)}.$$

If  $f'(x, z)$  and  $f''(y, z)$  are the components of  $f(x, y, z)$  which do not depend on  $y$  and  $x$ , respectively, we can recover the degree  $d_n$  of  $p_n(y, z)$  as the quotient  $d_n = \deg_x f' / \deg_y f''$ . Now the problem is to recover  $q_n(y, z)$  and  $\bar{h}(y, z)$ . Since  $\bar{h}(y, z)$  does not depend on  $x$ , we have that

$$\bar{h}(y, z) = \overline{h(x, y, z)} = \overline{h(0, y, z)}.$$

From the equality (7) and the condition  $p_n(0, z) = 0$  we obtain that

$$f(x, 0, z) = h(0, x + p_n(0, z), z) = h(0, x, z).$$

Hence  $h(0, y, z) = f(y, 0, z)$  and we are able to find  $\bar{h}(y, z)$ . We write  $\bar{h}$  and  $\bar{q}_n$  in the form

$$\bar{h}(y, z) = \theta(t_0, t_1, \dots, t_k) * y^k, \quad q_n(y, z) = \omega(t_0, t_1, \dots, t_d) * y^d,$$

where  $\theta(t_0, t_1, \dots, t_k) \in K[t_0, t_1, \dots, t_k]$  is known explicitly and  $\omega(t_0, t_1, \dots, t_d) \in K[t_0, t_1, \dots, t_d]$ . Similarly, the part of the component of maximal bidegree of  $f(x, y, z)$  which does not depend on  $x$  has the form

$$\overline{f''}(y, z) = \zeta(t_0, t_1, \dots, t_{kd}) * y^{kd}, \quad \zeta(t_0, t_1, \dots, t_{kd}) \in K[t_0, t_1, \dots, t_{kd}].$$

Since  $\bar{h}(q_n(y, z), z) = \overline{f''}(y, z)$ , by Lemma 2.1 we obtain

$$\begin{aligned} \zeta(t_0, t_1, \dots, t_{kd}) &= \theta(t_0, t_d, t_{2d}, \dots, t_{kd}) \omega(t_0, t_1, \dots, t_d) \\ &\quad \omega(t_d, t_{d+1}, \dots, t_{2d}) \cdots \omega(t_{(k-1)d}, t_{(k-1)d+1}, \dots, t_{kd}). \end{aligned}$$

Here we know  $\zeta$  and  $\theta$  and want to determine  $\omega$ . Let

$$\begin{aligned} \zeta'(t_0, t_1, \dots, t_{kd}) &= \zeta(t_0, t_1, \dots, t_{kd}) / \theta(t_0, t_d, t_{2d}, \dots, t_{kd}) \\ &= \omega(t_0, t_1, \dots, t_d) \omega(t_d, t_{d+1}, \dots, t_{2d}) \cdots \omega(t_{(k-1)d}, t_{(k-1)d+1}, \dots, t_{kd}). \end{aligned}$$

The greatest common divisor of the polynomials  $\zeta'(t_0, t_1, \dots, t_{kd})$  and  $\zeta'(t_{(k-1)d}, t_{(k-1)d+1}, \dots, t_{(2k-1)d})$  in  $K[t_0, t_1, \dots, t_{(2k-1)d}]$  is equal, up to a multiplicative constant  $\beta$ , to  $\omega(t_{(k-1)d}, t_{(k-1)d+1}, \dots, t_{kd})$ . Hence the knowledge of  $\zeta'$  allows to determine  $\beta\omega(t_0, t_1, \dots, t_k)$  as well as the value of  $\beta^d$ . This means that we know also all the possible values of  $\beta$  and the polynomial  $q_n(y, z)$ . Now we apply on  $f(x, y, z)$  the  $z$ -automorphism  $\sigma = (x - q_n(y, z), y)$ . Since  $f(x, y, z) - \bar{h}(x + q_n(y, z), z)$  is lower in the  $(d_n, 1)$ -biordering than  $f(x, y, z)$  itself, we may replace  $f$  with  $\sigma(f)$  and to make the next step. The considerations are almost the same when some of the automorphisms  $\rho_0$  and  $\rho_n$  is affine. For example, if  $f = \varphi(x)$  and  $\rho_n = (x + \gamma y, y)$ ,  $\gamma \in K$ , in (2), then the leading bihomogeneous component of  $h = \tau\rho_n^{-1}(f)$  does not depend on  $y$ , and we can do the next step. If  $f$  is a  $z$ -tame coordinate, then the above process will stop when we reduce  $f$  to a polynomial in the form  $\alpha x + p(y, z)$ . If  $f$  is not

a  $z$ -tame coordinate, then the process will also stop by different reason. In some step we shall reduce  $f(x, y, z)$  to a polynomial  $f_1(x, y, z)$ . It may turn out that the degree  $d = \deg_x f_1(x, 0, z) / \deg_y f_1(0, y, z)$  is not integer. Or, the commutative polynomials  $\theta$  and  $\omega$  corresponding to  $f_1$  do not exist.  $\square$

The following corollary is stronger than Corollary 2.4.

**Corollary 2.6.** *Let  $h(t, z) \in K\langle t, z \rangle$  and let  $\deg_u h(u, z) > 0$ . Then  $f(x, y, z) = x + h(xz - zy, z)$  is not a  $z$ -tame coordinate of  $K\langle x, y, z \rangle$ .*

*Proof.* We apply the algorithm in the proof of Theorem 2.5. Let  $f(x, y, z)$  be a  $z$ -tame coordinate and let  $h'(x, z) = h(xz, z)$  and  $h''(y, z) = h(-zy, z)$  be the polynomials obtained from  $h(xz - zy, z)$  replacing, respectively,  $y$  and  $x$  by 0. Clearly,  $\text{bideg}_x h' = \text{bideg}_y h''$ . Hence, as in the proof of Theorem 2.5 we can replace  $f(x, y, z)$  with  $\sigma(f)$ , where  $\sigma = (x - \alpha y, y)$ , for a suitable  $\alpha \in K^*$ , and the leading bihomogeneous component of  $\sigma(f)$  in the  $(1, 1)$ -ordering does not depend on  $y$ . But this brings to a contradiction. If  $h_1(t, z) \in K\langle t, z \rangle$  is homogeneous with respect to  $t$ , and

$$h_1((x - \gamma y)z - zy, z) = h_2(x, z)$$

for some  $h_2(x, z)$ , then, replacing  $x$  with 0, we obtain  $h_1(-(\gamma yz + zy), z) = 0$ , which is impossible.  $\square$

**Remark 2.7.** In Corollary 2.6, we cannot guarantee that the polynomial  $f(x, y, z) = x + h(xz - zy, z)$  is a  $z$ -coordinate at all. For example, let  $f(x, y, z) = x + (xz - zy)$  be a  $z$ -coordinate with a coordinate mate  $g(x, y, z)$ . If  $g_1(x, y, z)$  is the linear in  $x, y$  component of  $g$ , then  $\varphi_1 = (f, g_1)$  is also a  $z$ -automorphism. Then, for suitable polynomials  $c, d \in K[z_1, z_2]$ , the matrix

$$J_z(\varphi_1) = \begin{pmatrix} 1 + z_2 & c(z_1, z_2) \\ -z_1 & d(z_1, z_2) \end{pmatrix}$$

is invertible. If we replace  $z_1$  with 0 in its determinant  $\det(J_z) = (1 + z_2)d(z_1, z_2) - z_1c(z_1, z_2)$  we obtain that  $(1 + z_2)d_2(0, z_2) \in K^*$  which is impossible.

## 3. ENDOMORPHISMS WHICH ARE NOT AUTOMORPHISMS

In this section we shall establish a  $z$ -analogue of the following proposition which is the main step of the proof of the theorem of Czerniakiewicz [Cz] and Makar-Limanov [ML1, ML2] for the tameness of the automorphisms of  $K\langle x, y \rangle$ .

**Proposition 3.1.** *Let  $\varphi = (x + u, y + v)$  be an endomorphism of  $K\langle x, y \rangle$ , where  $u, v$  are in the commutator ideal of  $K\langle x, y \rangle$  and at least one of them is different from 0. Then  $\varphi$  is not an automorphism of  $K\langle x, y \rangle$ .*

An essential moment in its proof, see the book by Cohn [C], is the following lemma.

**Lemma 3.2.** *If  $f, g \in K\langle x, y \rangle$  are two bihomogeneous polynomials, then they either generate a free subalgebra of  $K\langle x, y \rangle$  or, up to multiplicative constants, both are powers of the same bihomogeneous element of  $K\langle x, y \rangle$ .*

We shall prove a weaker version of the lemma for  $K\langle x, y, z \rangle$  which will be sufficient for our purposes.

**Lemma 3.3.** *Let  $(0, 0) \neq (a, b) \in \mathbb{Z}^2$  and let  $f_1, f_2 \in K\langle x, y, z \rangle$  be bihomogeneous with respect to the  $(a, b)$ -degree of  $K\langle x, y, z \rangle$ , i.e.,  $\text{adeg}_x w + \text{bdeg}_y w$  is the same for all monomials of  $f_1$ , and similarly for  $f_2$ . If  $f_1$  and  $f_2$  are algebraically dependent, then both  $\deg_{(a,b)} f_1$  and  $\deg_{(a,b)} f_2$  are either nonnegative or nonpositive.*

*Proof.* Let  $v(f_1, f_2, z) = 0$  for some nonzero polynomial  $v(u_1, u_2, z) \in K\langle u_1, u_2, z \rangle$ . We may assume that both  $f_1, f_2$  depend not on  $z$  only. We fix a term-ordering on  $K\langle x, y, z \rangle$ . Let  $\tilde{f}_1$  and  $\tilde{f}_2$  be the leading monomials of  $f_1$  and  $f_2$ , respectively. For each monomial  $z^{k_0} u_{i_1} z^{k_1} \dots z^{k_{s-1}} u_{i_s} z^{k_s} \in K\langle u_1, u_2, z \rangle$  the leading monomial of  $z^{k_0} f_{i_1} z^{k_1} \dots z^{k_{s-1}} f_{i_s} z^{k_s} \in K\langle x, y, z \rangle$  is  $z^{k_0} \tilde{f}_{i_1} z^{k_1} \dots z^{k_{s-1}} \tilde{f}_{i_s} z^{k_s}$ . Hence, the algebraic dependence of  $f_1$  and  $f_2$  implies that two different monomials  $z^{k_0} \tilde{f}_{i_1} z^{k_1} \dots z^{k_{s-1}} \tilde{f}_{i_s} z^{k_s}$  and  $z^{l_0} \tilde{f}_{j_1} z^{l_1} \dots z^{l_{t-1}} \tilde{f}_{j_t} z^{l_t}$  are equal. We write  $\tilde{f}_1 = z^{p_1} g_1 z^{q_1}$  and  $\tilde{f}_2 = z^{p_2} g_2 z^{q_2}$ , where  $g_1, g_2$  do not start and do not end with  $z$ . After some cancelation in the equation

$$z^{k_0} \tilde{f}_{i_1} z^{k_1} \dots z^{k_{s-1}} \tilde{f}_{i_s} z^{k_s} = z^{l_0} \tilde{f}_{j_1} z^{l_1} \dots z^{l_{t-1}} \tilde{f}_{j_t} z^{l_t}$$

we obtain a relation of the form

$$(8) \quad g_{a_1} z^{m_1} \dots z^{m_{k-1}} g_{a_k} z^{m_k} = g_{b_1} z^{n_1} \dots z^{n_{l-1}} g_{b_l} z^{n_l},$$

with different  $g_{a_1}$  and  $g_{b_1}$ . Hence, if  $\deg g_1 \geq \deg g_2$ , then  $g_1 = g_2 g_3$  for some monomial  $g_3$  (and  $g_2 = g_1 g_3$  if  $\deg g_1 < \deg g_2$ ). Again,  $g_2$  and  $g_3$  satisfy a relation of the form (8). Since  $\deg g_1 \geq \deg g_2 > 0$ , we obtain  $\deg g_1 + \deg g_2 > \deg g_1 = \deg g_2 + \deg g_3$ . Applying inductive arguments, we derive that both  $\deg_{(a,b)} g_2$  and  $\deg_{(a,b)} g_3$  are either nonnegative or nonpositive, and the same holds for  $f_1$  and  $f_2$  because  $g_1 = g_2 g_3$ ,  $\deg_{(a,b)} g_1 = \deg_{(a,b)} g_2 + \deg_{(a,b)} g_3$ , and  $\deg_{(a,b)} f_i = \deg_{(a,b)} g_i$ ,  $i = 1, 2$ .  $\square$

The condition that  $u(x, y)$  and  $v(x, y)$  belong to the commutator ideal of  $K\langle x, y \rangle$ , as in Proposition 3.1, immediately implies that all monomials of  $u$  and  $v$  depend on both  $x$  and  $y$ , as required in the following theorem.

**Theorem 3.4.** *The  $z$ -endomorphisms of the form*

$$\varphi = (x + u(x, y, z), y + v(x, y, z)),$$

where  $(u, v) \neq (0, 0)$  and all monomials of  $u$  and  $v$  depend on both  $x$  and  $y$ , are not automorphisms of  $K\langle x, y, z \rangle$ .

*Proof.* The key moment in the proof of Proposition 3.1 is the following. If  $\varphi = (x + u, y + v)$  is an endomorphism of  $K\langle x, y \rangle$ , where  $u, v$  are in the commutator ideal of  $K\langle x, y \rangle$  and at least one of them is different from 0, then there exist two integers  $a$  and  $b$  such that  $(a, b) \neq (0, 0)$  and  $a \leq 0 \leq b$  with the property that  $\deg_{(a,b)}(x + u) = \deg_{(a,b)} x = a$  and  $\deg_{(a,b)}(y + v) = \deg_{(a,b)} y = b$ . Ordering in a suitable way the  $(a, b)$ -bidegrees, one concludes that the  $(a, b)$ -degrees of the leading bi-homogeneous components of  $x + u$  and  $y + v$  are with different signs. Then Lemma 3.2 shows that these leading components are algebraically independent and bidegree arguments as in the proof of Proposition 3.1 give that  $\varphi$  cannot be an automorphism. We repeat verbatim these arguments, working with the same  $(a, b)$ -(bi)degree and bidegree ordering Proposition 3.1, without counting the degree of  $z$ . In the final step, we use Lemma 3.3 instead of Lemma 3.2.  $\square$

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