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Frequency of Oscillations of an Error Term related to the Euler function

Y.-K. LAU and Y.-F.S. Pétermann

Abstract

Let $\phi$ be the Euler function, and consider the error term $H$ in the asymptotic formula

$$\sum_{n \leq x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + H(x).$$

We prove that for any fixed real number $A$, there are at least $C_A T + O(1)$ integers $n \in [1, T]$ such that $(H(n) - A)(H(n + 1) - A) < 0$, where $0 < C_A < 1$ is a constant depending on $A$. 
Let \( \phi \) be the Euler function (i.e. \( \phi(n) \) denotes the number of integers less than \( n \) which are relatively prime to \( n \)), and define
\[
H(x) = \sum_{n \leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2} x.
\]

In [2], it is shown that \( H(x) \) has a large number (of order \( T \)) of sign changes on integers \( n \leq T \). In this note, we prove that this phenomenon occurs as well for the changes in sign of \( H_A(n) = H(n) - A \), where \( A \) is any fixed real number. The value \( A = 3/\pi^2 \) plays a special role. It is indeed known that the distribution function \( \Delta \) of the values taken by \( H_3/\pi^2 \) at integers is symmetric [3], whence in particular \( \Delta(0) = 1/2 \): so one would expect the number of changes in sign of \( H_A(n) \) to be particularly important when \( A = 3/\pi^2 \). But the slightly surprising fact is that the only value of \( A \) for which a straightforward modification of the argument in [2] is uneficient is precisely \( A = 3/\pi^2 \).

**Theorem.** Let \( A \) be a fixed number. For all sufficiently large \( T \), we have
\[
|\{ n \in [1, T] : (H(n) - A)(H(n + 1) - A) < 0 \}| \geq C_A T
\]
where \(|\{\cdots\}|\) denotes the cardinality of the set and \( 0 < C_A < 1 \) is a constant (depending on \( A \)).

We separate the proof into three cases: (i) \( A < 3/\pi^2 \), (ii) \( A = 3/\pi^2 \) and (iii) \( A > 3/\pi^2 \). Cases (i) and (iii) can be treated as in [2], §3. For case (i) replace in the argument there \( D(0) \) by \( D(A) \) where \( D(u) = \lim_{x \to \infty} x^{-1} |\{ n \leq x : H(n) \leq u \}| \), and note that if \( H(n) < A \) and \( H(m) < A \) for all integral \( m \in [n, n + 2h) \), then for any real \( t \in [n, n + h) \) we have
\[
\left| \int_t^{t+h} H(u) \, du \right| \geq \left( \frac{3}{\pi^2} - A \right) (h - 2),
\]
as soon as \( h \) is large enough. This comes from the fact that \( H(x) \) is a straight line of slope \(-6/\pi^2\) in every interval \([m, m + 1)\) when \( m \) is an integer. For case (iii) consider instead the proportion \((1 - D(A))\) of integers \( n \) for which \( H(n) > A \), and similarly note that if \( H(n) > A \) and \( H(m) > A \) for all integral \( m \in [n, n + 2h) \), then for any real \( t \in [n, n + h) \) we have
\[
\left| \int_t^{t+h} H(u) \, du \right| \geq \left( A - \frac{3}{\pi^2} \right) (h - 2),
\]
as soon as $h$ is large enough. It is now clear why this method doesn’t work when $A = 3/\pi^2$.

From [1] and [3], we know that the distribution function $D(u)$ exists, $D(3/\pi^2) = 1/2$ and $D(u)$ is a continuous function of $u$. Hence, for all sufficiently large $T$, we have

$$|\{T \leq n \leq 2T : H(n) \leq 3/\pi^2\}| \geq \frac{3T}{T}.$$

Let $h$ be a large parameter, which will be chosen later. We divide the interval $[T, 2T]$ into divisions of length $h$, and group every 8 divisions to form an interval. Then the number of these newly formed intervals is $\lceil T/(8h) \rceil$, which is at most $T/(7h)$ for all sufficiently large $T$. For convenience, we use the symbol $I$ to designate a subinterval of $I$ consisting of the initial 6 divisions. Define

$$C = \{I : H(n) \leq 3/\pi^2 \text{ for some } n \in I\}.$$

By (1), $|C| \geq (3T/7 - (2h) \times T/(7h))/(6h) = T/(42h)$. From the continuity of $D(u)$, we can find $\epsilon > 0$ such that the set $S = \{n \leq 2T : 3/\pi^2 - \epsilon \leq H(n) \leq 3/\pi^2\}$ has cardinality $|S| \leq T/168$. Consider $J_1 = \{I \in C : |I \cap S| \leq h/2\}$. Then

$$\frac{h}{2} |C \setminus J_1| \leq \sum_{I \in C \setminus J_1} |I \cap S| \leq |S| \leq \frac{T}{168}.$$

From this, we have $|J_1| \geq T/(100h)$. Then we can proceed with the argument in [2] on the collection $J_1$. Define

$$J_2 = \{I \in J_1 : H(m) \leq 3/\pi^2 \text{ for all integers } m \in [n, n+h] \text{ where } n \in I\}.$$

As $I \in J_2$ has at most $h/2$ elements in $S$ and $H(m) < 3/\pi^2 - \epsilon$ if $m \notin S$, we have

$$\epsilon^2 h^3 \ll \sum_{I \in J_2} \int_{n}^{n+h} \left( \int_{t}^{t+h} H(u) \, du \right)^2 \, dt \leq \int_{T}^{2T} \left( \int_{t}^{t+h} H(u) \, du \right)^2 \, dt$$

where the implied constants are independent of $\epsilon$ and $h$. The first inequality comes again from the fact that $H(x)$ is a straight line in every interval $[m, m+1)$ when $m$ is an integer. But the last integral is $\ll Th$ by [2, Main Lemma]. Thus,

$$|J_1 \setminus J_2| > \frac{T}{100h} - O\left( \frac{T}{\epsilon^2 h^2} \right).$$

Our assertion follows by taking $h$ to be a sufficiently large constant.

**Last Remark:** This method can be applied to the error term

$$E(x) = \sum_{n \leq x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6} x + \frac{1}{2} \log x$$
associated with the sum-of-divisors function $\sigma$ as well. In this case the critical value for which the argument of case (ii) applies is $A = \pi^2/12$.

References


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