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<th>Frequency of oscillations of an error term related to the Euler function</th>
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Abstract

Let $\phi$ be the Euler function, and consider the error term $H$ in the asymptotic formula

$$\sum_{n \leq x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + H(x).$$

We prove that for any fixed real number $A$, there are at least $C_A T + O(1)$ integers $n \in [1, T]$ such that $(H(n) - A)(H(n + 1) - A) < 0$, where $0 < C_A < 1$ is a constant depending on $A$. 

Let $\phi$ be the Euler function (i.e. $\phi(n)$ denotes the number of integers less than $n$ which are relatively prime to $n$), and define

$$H(x) = \sum_{n \leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2} x.$$

In [2], it is shown that $H(x)$ has a large number (of order $T$) of sign changes on integers $n \leq T$. In this note, we prove that this phenomenon occurs as well for the changes in sign of $H_A(n) = H(n) - A$, where $A$ is any fixed real number. The value $A = 3/\pi^2$ plays a special role. It is indeed known that the distribution function $\Delta$ of the values taken by $H_{3/\pi^2}$ at integers is symmetric [3], whence in particular $\Delta(0) = 1/2$: so one would expect the number of changes in sign of $H_A(n)$ to be particularly important when $A = 3/\pi^2$. But the slightly surprising fact is that the only value of $A$ for which a straightforward modification of the argument in [2] is inefficient is precisely $A = 3/\pi^2$.

**Theorem** Let $A$ be a fixed number. For all sufficiently large $T$, we have

$$|\{n \in [1, T] : (H(n) - A)(H(n + 1) - A) < 0\}| \geq C_A T$$

where $|\{\cdot\cdot\cdot\}|$ denotes the cardinality of the set and $0 < C_A < 1$ is a constant (depending on $A$).

We separate the proof into three cases: (i) $A < 3/\pi^2$, (ii) $A = 3/\pi^2$ and (iii) $A > 3/\pi^2$. Cases (i) and (iii) can be treated as in [2], §3. For case (i) replace in the argument there $D(0)$ by $D(A)$ where $D(u) = \lim_{x \to \infty} x^{-1}|\{n \leq x : H(n) \leq u\}|$, and note that if $H(n) < A$ and $H(m) < A$ for all integral $m \in [n, n + 2h)$, then for any real $t \in [n, n + h)$ we have

$$\left|\int_t^{t+h} H(u) \, du\right| \geq (\frac{3}{\pi^2} - A)(h - 2),$$

as soon as $h$ is large enough. This comes from the fact that $H(x)$ is a straight line of slope $-6/\pi^2$ in every interval $[m, m + 1)$ when $m$ is an integer. For case (iii) consider instead the proportion $(1 - D(A))$ of integers $n$ for which $H(n) > A$, and similarly note that if $H(n) > A$ and $H(m) > A$ for all integral $m \in [n, n + 2h)$, then for any real $t \in [n, n + h)$ we have

$$\left|\int_t^{t+h} H(u) \, du\right| \geq (A - \frac{3}{\pi^2})(h - 2),$$

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as soon as \( h \) is large enough. It is now clear why this method doesn’t work when 
\( A = 3/\pi^2 \).

From [1] and [3], we know that the distribution function \( D(u) \) exists, \( D(3/\pi^2) = 1/2 \) and \( D(u) \) is a continuous function of \( u \). Hence, for all sufficiently large \( T \), we have

\[
|\{T \leq n \leq 2T : H(n) \leq 3/\pi^2\}| \geq \frac{3T}{T}.
\]  

(1)

Let \( h \) be a large parameter, which will be chosen later. We divide the interval [\( T, 2T \)] into divisions of length \( h \), and group every 8 divisions to form an interval. Then the number of these newly formed intervals is \( [T/(8h)] \), which is at most \( T/(7h) \) for all sufficiently large \( T \). For convenience, we use the symbol \( I \) to designate a subinterval of \( I \) consisting of the initial 6 divisions. Define

\[
\mathcal{C} = \{I : H(n) \leq 3/\pi^2 \text{ for some } n \in I\}.
\]

By (1), \( |\mathcal{C}| \geq (3T/7 - (2h) \times T/(7h))/(6h) = T/(42h) \). From the continuity of \( D(u) \), we can find \( \epsilon > 0 \) such that the set \( S = \{n \leq 2T : 3/\pi^2 - \epsilon \leq H(n) \leq 3/\pi^2\} \) has cardinality \( |S| \leq T/168 \). Consider \( J_1 = \{I \in \mathcal{C} : |I \cap S| \leq h/2\} \). Then

\[
\frac{h}{2} |\mathcal{C} \setminus J_1| \leq \sum_{I \in \mathcal{C} \setminus J_1} |I \cap S| \leq |S| \leq \frac{T}{168}.
\]

From this, we have \( |J_1| \geq T/(100h) \). Then we can proceed with the argument in [2] on the collection \( J_1 \). Define

\[
J_2 = \{I \in J_1 : H(m) \leq 3/\pi^2 \text{ for all integers } m \in [n, n+h] \text{ where } n \in I\}.
\]

As \( I \in J_2 \) has at most \( h/2 \) elements in \( S \) and \( H(m) < 3/\pi^2 - \epsilon \) if \( m \notin S \), we have

\[
\epsilon^2 h^3 \ll \sum_{I \in J_2} \int_n^{n+h} \left( \int_t^{t+h} H(u) \, du \right)^2 \, dt \leq \int_T^{2T} \left( \int_t^{t+h} H(u) \, du \right)^2 \, dt
\]

where the implied constants are independent of \( \epsilon \) and \( h \). The first inequality comes again from the fact that \( H(x) \) is a straight line in every interval \( [m, m+1] \) when \( m \) is an integer. But the last integral is \( \ll Th \) by [2, Main Lemma]. Thus,

\[
|J_1 \setminus J_2| > \frac{T}{100h} - O\left( \frac{T}{\epsilon^2 h^2}\right).
\]

Our assertion follows by taking \( h \) to be a sufficiently large constant.

**Last Remark:** This method can be applied to the error term

\[
E(x) = \sum_{n \leq x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6} x + \frac{1}{2} \log x
\]
associated with the sum-of-divisors function $\sigma$ as well. In this case the critical value for which the argument of case (ii) applies is $A = \pi^2/12$.

References


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