

# Coordinates on Schubert cells, Kostant's harmonic forms, and the Bruhat-Poisson structure on $G/B$

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## 1 Introduction and Notation

This work grew out of my attempts to understand the relations between results of Kostant [K] on the de Rham cohomology of a flag manifold and Poisson geometry.

Let  $X = G/B$  be a flag manifold, where  $G$  is a complex semi-simple Lie group and  $B$  is a Borel subgroup of  $G$ . Let  $K$  be a compact real form of  $G$ , so  $X = G/B \cong K/T$ , where  $T = K \cap B$  is a maximal torus of  $K$ . In [K], Kostant constructs, for each element  $w$  in the Weyl group  $W$ , an explicit  $K$ -invariant closed differential form  $s^w$  on  $X$  with  $\deg(s^w) = 2l(w)$ , where  $l(w)$  is the length of  $w$  (see Section 4). The cohomology classes of the  $s^w$ 's form a basis of  $H(X, \mathbb{C})$  that, up to constant multiples, is dual to the basis of the homology of  $X$  formed by the closures of the Bruhat (or Schubert) cells in  $X$ . These  $K$ -invariant forms on  $X$  are  $(d, \partial)$ -harmonic, where  $d$  is the de Rham differential operator and  $\partial$  is a degree  $-1$  operator introduced by Kostant.

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Our work was first motivated by wanting to understand the nature of the operator  $\partial$  in terms of Poisson geometry.

The Poisson structure that is relevant to Kostant's theorem is the so-called Bruhat-Poisson structure on  $X$  [L-W]. It has its origin in the theory of quantum groups. It has the special property that its symplectic leaves are precisely the Bruhat cells in  $X$  (and hence the name). More properties of the Bruhat-Poisson structure are reviewed in Section 3.

The Bruhat-Poisson structure gives rise to the degree  $-1$  operator

$$\partial_\pi = i_\pi d - di_\pi$$

on the space of differential forms on  $X$  called the Koszul-Brylinski operator [Ko] [B], where  $\pi$  is the bi-vector field on  $X$  defining this Poisson structure, and  $i_\pi$  denotes the contraction operator of differential forms with  $\pi$ . It satisfies  $\partial_\pi^2 = 0$ . Its homology is called the Poisson homology of  $\pi$ . Poisson homology is closely related to cyclic homology of associative algebras, as is shown in [B]. It turns out that when restricted to  $K$ -invariant differential forms on  $X$ , Kostant's operator  $\partial$  and the Koszul-Brylinski operator  $\partial_\pi$  differ by the contraction operator by a vector field  $\theta_0$  called the modular vector field of  $\pi$  (see [W2] [B-Z] [E-L-W]). More explicitly, it is the infinitesimal generator of the  $K$ -action on  $X$  in the direction of the element  $2iH_\rho$  with  $\rho$  being half of the sum of all positive roots. This result, together with others on the Poisson (co)homology of the Bruhat-Poisson structure, can be found in [E-L].

In this paper, we relate Kostant's harmonic forms on  $X$  with the Bruhat-Poisson structure.

More precisely, the Bruhat-Poisson structure on  $X = K/T$  is  $T$ -invariant (but not  $K$ -invariant). Thus, each Bruhat cell  $\Sigma_w$  inherits a  $T$ -invariant symplectic structure  $\Omega_w$ . Use  $\phi_w : \Sigma_w \rightarrow \mathfrak{t}^*$  to denote the moment map and let  $\mu_w$  be the Liouville volume form on  $\Sigma_w$  defined by  $\Omega_w$ . Theorem 4.3 says that when restricted to the cell  $\Sigma_w$ , Kostant's form  $s^w$  is related to the Liouville volume form  $\mu_w$  by

$$s^w|_{\Sigma_w} = e^{\langle \phi_w, 2iH_\rho \rangle} \mu_w.$$

Notice that the function  $\langle \phi_w, 2iH_\rho \rangle$  is the Hamiltonian function for the modular vector field  $\theta_0$  on  $\Sigma_w$ . (The vector field  $\theta_0$  is not globally Hamiltonian (see Section 4), but it is on each cell). Theorem 4.3 thus expresses Kostant's harmonic forms totally in terms of data coming from the Bruhat-Poisson structure. In particular, it shows that the integral  $\int_{\Sigma_w} s^w$  is of the Duistermaat - Heckman type. Wanting to see this was another motivation for this work.

Theorem 4.3 is proved by writing everything down in some coordinates

$$\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$$

on each Bruhat cell  $\Sigma_w$ , where  $l = l(w)$ . These coordinates are motivated by, but are independent of, the Bruhat-Poisson structure. Among the quantities that we write down explicitly in the coordinates are (see the later sections for the notation)

- (Theorem 3.4) the symplectic 2-form  $\Omega_w$  on  $\Sigma_w$  and thus the Liouville volume form  $\mu_w$ :

$$\begin{aligned}\Omega_w &= \sum_{j=1}^l \frac{i}{\ll \alpha_j, \alpha_j \gg} \frac{1}{1 + |z_j|^2} dz_j \wedge d\bar{z}_j \\ \mu_w &= \prod_{j=1}^l \frac{i}{\ll \alpha_j, \alpha_j \gg} \frac{1}{1 + |z_j|^2} dz_j \wedge d\bar{z}_j\end{aligned}$$

- (Theorem 3.4) the moment map for the  $T$ -action on  $(\Sigma_w, \Omega_w)$ :

$$\phi_w : \Sigma_w \longrightarrow \mathfrak{t}^* : \quad \phi_w = \sum_{j=1}^l \left( -\frac{1}{2} \log(1 + |z_j|^2) \check{H}_{\alpha_j} \right)$$

- (Theorem 2.7) the Haar measure  $dn$  on the group  $N_w = N \cap wN_-w^{-1}$  that parametrizes  $\Sigma_w$ :

$$dn = \left( \prod_{j=1}^l \frac{i \ll \rho, \beta_j \gg}{\pi \ll \beta_j, \beta_j \gg} \right) \prod_{j=1}^l (1 + |z_j|^2)^{\frac{2 \ll \rho, \beta_j \gg}{\ll \beta_j, \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j$$

- (Theorem 2.5) the  $A$ -component  $a_w(n)$  in the Iwasawa decomposition of the element  $\dot{w}^{-1}n\dot{w}$  for  $n \in N_w$  (where  $\dot{w}$  is any representative of  $w$  in  $K$ ):

$$a_w(n) = \prod_{j=1}^l \exp\left(\frac{1}{2} \log(1 + |z_j|^2) \check{H}_{\beta_j}\right)$$

- (Theorem 4.3) Kostant's harmonic form  $s^w$  restricted to  $\Sigma_w$ :

$$s^w|_{\Sigma_w} = \prod_{j=1}^l \frac{i}{\ll \alpha_j, \alpha_j \gg} (1 + |z_j|^2)^{-\frac{2 \ll \rho, \alpha_j \gg}{\ll \alpha_j, \alpha_j \gg} - 1} dz_j \wedge d\bar{z}_j.$$

By comparing these explicit formulas, we immediately get (Corollaries 3.5 and 3.6)

$$\begin{aligned}\phi_w &= Ad_{\dot{w}} \log a_w(n) \\ \mu_w &= \left( \prod_{j=1}^l \frac{\pi}{\ll \rho, \beta_j \gg} \right) a_w(n)^{-2\rho} dn.\end{aligned}$$

These relate the moment map  $\phi_w$  and the Liouville volume form  $\mu_w$  to the familiar map  $a_w$  and the Haar measure  $dn$  on  $N_w$ . This is desirable for understanding the geometry of the Bruhat-Poisson structure. The relation between the differential form  $s^w|_{\Sigma_w}$  and the Liouville form  $\mu_w$  also follows immediately from these formulas.

The coordinates  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  are presented first in Section 2. Here we derive the formulas for the Haar measure  $dn$  of  $N_w$  and for the map  $a_w : N \rightarrow A$ . As one other application of these coordinates, we show that Harish-Chandra's formula for the  $c$ -function (in the case of a complex group) follows easily as a product of 1-dimensional integrals. Our calculation here is easier because we have pushed the usual induction argument on the length of  $w$  into the calculations for the explicit formulas for  $a_w(n)$  and  $dn$ . The total amount of effort is probably the same.

In Section 3, we review some of the properties of the Bruhat-Poisson structure on  $K/T$ . We derive the formulas for the symplectic form  $\Omega_w$ , thus also the Liouville measure  $\mu_w$ , and the moment map for the  $T$  action on each Bruhat cell in the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$ -coordinates. By using the formulas for  $a_w(n)$  and  $dn$  given in Section 2, we arrive at the (coordinate-free) interpretations for both the moment map  $\phi_w$  and the Liouville measure  $\mu_w$  in terms of  $a_w(n)$  and  $dn$  as given in Corollaries 3.5 and 3.6.

Kostant's harmonic forms are reviewed in Section 4. Theorem 4.3 is given as an easy corollary of our formulas in the  $z$ -coordinates. Applications of Theorem 4.3 to the calculations of the Poisson (co)homology of the Bruhat-Poisson structure are given in [E-L].

In the Appendix, we discuss how our coordinates  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  are related to the (complex) Bott-Samelson coordinates.

We now fix the notation.

Let  $G$  be a finite dimensional complex semi-simple Lie group with Lie algebra  $\mathfrak{g}$ . Let  $H$  be a Cartan subgroup and  $\mathfrak{h}$  its Lie subalgebra. Denote by  $R$  the set of all roots of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ . Let  $R^+$  be a choice of positive roots. We will also write  $\alpha > 0$  for  $\alpha \in R^+$ . Let  $\mathfrak{b} = \mathfrak{b}_+$  be the Borel subalgebra spanned by  $\mathfrak{h}$  and all the positive root vectors, and let  $B$  be the corresponding Borel subgroup.

Let  $W$  be the Weyl group of  $G$  relative to  $H$ . The Bruhat decomposition

$$G = \bigcup_{w \in W} B\dot{w}B,$$

where  $\dot{w}$  is a representative of  $w$  in the normalizer of  $H$  in  $G$ , gives rise to the Bruhat decomposition

$$G/B = \bigcup_{w \in W} B\dot{w}B/B$$

of the flag manifold  $G/B$  into a disjoint union of cells. These cells, denoted by  $\Sigma_w = B\dot{w}B/B$ , are called Bruhat or Schubert cells, and their closures,  $X_w = \overline{\Sigma_w}$ , are called the Schubert varieties.

We choose a compact real form  $\mathfrak{k}$  of  $\mathfrak{g}$  as follows: let  $\ll \gg$  be the Killing form of  $\mathfrak{g}$ . For each positive root  $\alpha$ , denote by  $H_\alpha$  the image of  $\alpha$  under the isomorphism  $\mathfrak{h}^* \rightarrow \mathfrak{h}$  via  $\ll \cdot, \cdot \gg$ , i.e., for any  $H \in \mathfrak{h}$ ,

$$\ll H_\alpha, H \gg = \alpha(H).$$

Choose root vectors  $E_\alpha$  and  $E_{-\alpha}$  for  $\alpha$  and  $-\alpha$  respectively such that  $\ll E_\alpha, E_{-\alpha} \gg = 1$ . Then  $[E_\alpha, E_{-\alpha}] = H_\alpha$ . Set

$$X_\alpha = E_\alpha - E_{-\alpha}, \quad Y_\alpha = i(E_\alpha + E_{-\alpha}). \quad (1)$$

The real subspace

$$\mathfrak{k} = \text{span}_{\mathbb{R}}\{iH_\alpha, X_\alpha, Y_\alpha : \alpha > 0\}$$

is a compact real form of  $\mathfrak{g}$ . Let  $K$  be the corresponding compact subgroup of  $G$ . The intersection  $T = K \cap B$  is a maximal torus of  $K$ , and its Lie algebra is

$$\mathfrak{t} = \text{span}_{\mathbb{R}}\{iH_\alpha : \alpha > 0\}.$$

Let  $\mathfrak{a} = i\mathfrak{t}$ , and let  $\mathfrak{n} = \mathfrak{n}_+$  be the subalgebra of  $\mathfrak{g}$  spanned by all the positive root vectors. Let  $A$  and  $N$  be the corresponding subgroups. Then

$$G = KAN$$

is the Iwasawa decomposition of  $G$  as a real semi-simple Lie group. It follows that the projection map from  $K \subset G$  to  $G/B$  induces an isomorphism from  $K/T$  to  $G/B$ . From now on, we will identify  $G/B$  with  $K/T$  this way.

For  $g \in G$ , let  $g = k_g a_g n_g$  be the Iwasawa decomposition of  $g$ . The map

$$G \times K \longrightarrow K : (g, k) \longmapsto k_{gk} \quad (2)$$

defines a left action of  $G$  on  $K$ . We will denote this action by  $(g, k) \rightarrow g \circ k$ .

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## 2 The coordinates $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$ on $N_w$

Let  $w \in W$  be a Weyl group element with length  $l = l(w)$ . Set

$$R_w^+ = \{\alpha > 0 : w^{-1}\alpha < 0\}$$

and

$$\mathfrak{n}_w = \text{span}_{\mathbb{C}}\{E_\alpha : \alpha \in R_w^+\},$$

and let  $N_w$  be the subgroup of  $G$  with Lie algebra  $\mathfrak{n}_w$ . Then  $N_w = N \cap wN_-w^{-1}$ , where  $N_-$  is the “opposite” of  $N$ , and the map

$$j_w : N_w \longrightarrow \Sigma_w : n \longmapsto nw/B \quad (3)$$

is a holomorphic diffeomorphism. Thus, any complex coordinate system on  $N_w$  will give one on  $\Sigma_w$ . The coordinates  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  that we will introduce in this section, however, will not be complex for a general  $w$  with  $l(w) > 1$  (see Example 2.9). To obtain these coordinates, we make use of the compact real form  $K$  of  $G$  and the isomorphism of  $G/B$  and  $K/T$ . They are motivated by the Bruhat-Poisson structure on  $K/T$ .

Let  $\dot{w}$  be a representative of  $w$  in  $K$ . For  $n \in N_w$ , let  $a_w(n)$  be the  $A$ -component of the element  $\dot{w}^{-1}n\dot{w}$  for the Iwasawa decomposition, i.e.,

$$\dot{w}^{-1}n\dot{w} = k_1 a_w(n) m_1 \quad (4)$$

for some  $k_1 \in K$  and  $m_1 \in N$ . Notice that the map  $a_w : N_w \rightarrow A$  depends only on  $w$  and not on the choice of  $\dot{w}$ .

In this section, we will write down the map  $a_w : N_w \rightarrow A$  explicitly in the coordinates  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  (Theorem 2.5). We will also write down the Haar measure of  $N_w$  in these coordinates (Theorem 2.7). As an application, we show how Harish-Chandra’s formula for the  $c$ -function also follows easily as a product of 1-dimensional integrals.

We now describe the coordinates  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$ .

Let again  $\dot{w}$  be a representative of  $w$  in  $K$ . Let

$$C_{\dot{w}} = N_w \circ \dot{w}$$

be the  $N_w$ -orbit in  $K$  through  $\dot{w}$  of the action  $(n, k) \rightarrow n \circ k$ . Recall that  $n \circ k = k_1$  if  $nk = k_1 b$  is the Iwasawa decomposition of  $nk$  with  $k_1 \in K$  and  $b \in AN$ . The map

$$J_{\dot{w}} : N_w \longrightarrow C_{\dot{w}} : n \longmapsto n \circ \dot{w}$$

is then a diffeomorphism. Moreover,  $J_{\dot{w}}$  followed by the projection map from  $C_{\dot{w}} \subset K \subset G$  to  $G/B$  is just the map  $j_w$  in (3). Thus  $C_{\dot{w}}$  is a lift of  $\Sigma_w$  in  $K$ .

**The case of  $l(w) = 1$ .** When  $w = \sigma_\gamma$  is the simple reflection corresponding to a simple root  $\gamma$ , we use, for notational simplicity,  $N_\gamma$  to denote  $N_{\sigma_\gamma}$  and  $\dot{\gamma}$  for  $\dot{\sigma}_\gamma$ . In this case, set

$$\check{H}_\gamma = \frac{2}{\ll \gamma, \gamma \gg} H_\gamma, \quad \check{E}_\gamma = \sqrt{\frac{2}{\ll \gamma, \gamma \gg}} E_\gamma, \quad \check{E}_{-\gamma} = \sqrt{\frac{2}{\ll \gamma, \gamma \gg}} E_{-\gamma}.$$

Then the map

$$\psi_\gamma : sl(2, \mathbb{C}) \longrightarrow \mathfrak{g} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \check{H}_\gamma, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \check{E}_\gamma, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \check{E}_{-\gamma} \quad (5)$$

is a Lie algebra homomorphism. It induces a Lie group homomorphism

$$\Psi_\gamma : SL(2, \mathbb{C}) \longrightarrow G,$$

and  $\Psi_\gamma(SU(2)) \subset K$ . For  $z \in \mathbb{C}$ , let

$$n_z = \Psi_\gamma \left( \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \right) = \exp(z \check{E}_\gamma) \in N_\gamma.$$

The map

$$\mathbb{C} \longrightarrow N_\gamma : z \longmapsto n_z$$

is clearly a parametrization of  $N_\gamma$  by  $\mathbb{C}$ . This is a complex coordinate system on  $N_\gamma$ .

Notice that the element  $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$  in  $SU(2)$  is a representative of the non-trivial element in the Weyl group of  $SL(2, \mathbb{C})$ . Let

$$\dot{\gamma} = \Psi_\gamma \left( \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right) \in K.$$

It is a representative of  $\sigma_\gamma$  in  $K$ . The following Iwasawa decomposition in  $SL(2, \mathbb{C})$

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} \frac{iz}{\sqrt{1+|z|^2}} & \frac{i}{\sqrt{1+|z|^2}} \\ \frac{i}{\sqrt{1+|z|^2}} & \frac{-i\bar{z}}{\sqrt{1+|z|^2}} \end{pmatrix} \begin{pmatrix} \sqrt{1+|z|^2} & \frac{\bar{z}}{\sqrt{1+|z|^2}} \\ 0 & \frac{1}{\sqrt{1+|z|^2}} \end{pmatrix}$$

gives the Iwasawa decomposition of  $n_z \dot{\gamma}$  in  $G$  as

$$n_z \dot{\gamma} = \Psi_\gamma \left( \begin{pmatrix} \frac{iz}{\sqrt{1+|z|^2}} & \frac{i}{\sqrt{1+|z|^2}} \\ \frac{i}{\sqrt{1+|z|^2}} & \frac{-i\bar{z}}{\sqrt{1+|z|^2}} \end{pmatrix} \right) \Psi_\gamma \left( \begin{pmatrix} \sqrt{1+|z|^2} & \frac{\bar{z}}{\sqrt{1+|z|^2}} \\ 0 & \frac{1}{\sqrt{1+|z|^2}} \end{pmatrix} \right).$$

Thus

$$n_z \circ \dot{\gamma} = \Psi_\gamma \left( \begin{array}{cc} \frac{iz}{\sqrt{1+|z|^2}} & \frac{i}{\sqrt{1+|z|^2}} \\ \frac{i}{\sqrt{1+|z|^2}} & \frac{-i\bar{z}}{\sqrt{1+|z|^2}} \end{array} \right) \in K.$$

The map

$$\mathbb{C} \longrightarrow C_{\dot{\gamma}} : z \longmapsto n_z \circ \dot{\gamma} = \Psi_\gamma \left( \begin{array}{cc} \frac{iz}{\sqrt{1+|z|^2}} & \frac{i}{\sqrt{1+|z|^2}} \\ \frac{i}{\sqrt{1+|z|^2}} & \frac{-i\bar{z}}{\sqrt{1+|z|^2}} \end{array} \right)$$

is a parametrization of  $C_{\dot{\gamma}}$  by  $\{z, \bar{z}\}$ . We also see that

$$a_{\sigma_\gamma}(n_z) = \exp\left(\frac{1}{2} \log(1+|z|^2) \check{H}_\gamma\right).$$

**The general case.** For a general element  $w \in W$ , let  $l = l(w)$  be the length of  $w$ , and let

$$w = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_l} \tag{6}$$

be a reduced decomposition, where  $\gamma_1, \gamma_2, \dots, \gamma_l$  are simple roots. Again for notational simplicity, we use  $N_{\gamma_j}$  to denote  $N_{\sigma_{\gamma_j}}$  for  $j = 1, \dots, l$ . We now have the Lie group homomorphism

$$\Psi_{\gamma_j} : SL(2, \mathbb{C}) \longrightarrow G$$

for each  $j$ . Let again

$$\dot{\gamma}_j = \Psi_{\gamma_j} \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right) \in K.$$

Write an element in  $N_{\gamma_j}$  as

$$n_{z_j} = \exp(z_j \check{E}_{\gamma_j})$$

for  $z_j \in \mathbb{C}$ . Set

$$\dot{w} = \dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_l \in K.$$

**Theorem 2.1** *There is a diffeomorphism (between real manifolds)*

$$F_w : N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l} \longrightarrow N_w$$

characterized by

$$F_w(n_1, n_2, \dots, n_l) \circ \dot{w} = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_l \circ \dot{\gamma}_l) \in K. \tag{7}$$

The map

$$\mathbb{C}^l \longrightarrow N_w : (z_1, z_2, \dots, z_l) \longmapsto F_w(n_{z_1}, n_{z_2}, \dots, n_{z_l}) \tag{8}$$

gives coordinates  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  on  $N_w$  (as a real manifold).

**Remark 2.2** This is the same as saying that the map

$$C_{\dot{\gamma}_1} \times C_{\dot{\gamma}_2} \times \cdots \times C_{\dot{\gamma}_l} \longrightarrow C_{\dot{w}}$$

given by multiplication in  $K$  is a diffeomorphism. This statement is given in [S]. The proof we give below contains a recursive formula for  $F_w$  that will be used later.

**Remark 2.3** We use  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  instead of  $\{z_1, z_2, \dots, z_l\}$  to denote the coordinates to emphasize the fact that the map in (8) is in general not a holomorphic diffeomorphism. See Example 2.9.

**Remark 2.4** Even though we use  $F_w$  to denote the map in Theorem 2.1, it depends not only on  $w$  but also on the reduced decomposition  $w = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_l}$  for  $w$ . Therefore, the coordinates  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  depend on the choice of the reduced decomposition.

**Proof of Theorem 2.1.** We need to show that for every point  $(n_1, \dots, n_l) \in N_{\gamma_1} \times \cdots \times N_{\gamma_l}$ , there exists (a necessarily unique)  $F(n_1, \dots, n_l) \in N_w$  such that (7) is satisfied. We also need to show that each  $n \in N_w$  arises this way. We prove this by induction on  $l(w)$ . When  $l(w) = 1$ , the map  $F_w$  is the identity map. Now for  $w$  with  $l(w) > 1$  and with the reduced decomposition given in (6), set

$$w_1 = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_{l-1}}$$

so that  $w = w_1 \sigma_{\gamma_l}$ . We wish to relate  $F_w$  and  $F_{w_1}$ . To this end, we first recall that  $N_{w_1} \subset N_w$  [J]. In fact, the multiplication map in  $N$  gives a diffeomorphism

$$N_{w_1} \times \dot{w}_1 N_{\gamma_l} \dot{w}_1^{-1} \longrightarrow N_w,$$

where

$$\dot{w}_1 = \dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{l-1}.$$

Now given  $(n_1, n_2, \dots, n_l) \in N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l}$ , let

$$n' = F_{w_1}(n_1, n_2, \dots, n_{l-1}) \in N_{w_1}$$

be such that

$$n' \circ (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{l-1}) = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_{l-1} \circ \dot{\gamma}_{l-1}) \in K.$$

We search for  $x \in \dot{w}_1 N_{\gamma_l} \dot{w}_1^{-1}$  such that  $n = n'x \in N_w$  satisfies

$$n \circ (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_l) = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_l \circ \dot{\gamma}_l) \in K. \quad (9)$$

Now

$$\begin{aligned} n \dot{\gamma}_1 \cdots \dot{\gamma}_l &= n'x \dot{\gamma}_1 \cdots \dot{\gamma}_{l-1} \dot{\gamma}_l \\ &= (n' \dot{\gamma}_1 \cdots \dot{\gamma}_{l-1}) (\dot{w}_1^{-1} x \dot{w}_1) \dot{\gamma}_l \\ &= (n' \dot{\gamma}_1 \cdots \dot{\gamma}_{l-1}) x' \dot{\gamma}_l, \end{aligned}$$

where

$$x' = \dot{w}_1^{-1} x \dot{w}_1 \in N_{\gamma_l}.$$

In the notation we have introduced so far, we have

$$n' \dot{\gamma}_1 \cdots \dot{\gamma}_{l-1} = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_{l-1} \circ \dot{\gamma}_{l-1}) a_{w_1}(n') m' \quad (10)$$

for some  $m' \in N$ . Thus

$$n \dot{\gamma}_1 \cdots \dot{\gamma}_l = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_{l-1} \circ \dot{\gamma}_{l-1}) a_{w_1}(n') m' x' \dot{\gamma}_l.$$

Denote by  $N_{\hat{\gamma}_l}$  the subgroup of  $N$  with Lie algebra

$$\mathfrak{n}_{\hat{\gamma}_l} = \text{span}_{\mathbb{C}}\{E_\alpha : \alpha > 0, \alpha \neq \gamma_l\}.$$

Then the multiplication map in  $N$  induces a diffeomorphism  $N_{\gamma_l} \times N_{\hat{\gamma}_l} \rightarrow N$ , so we have the decomposition  $N = N_{\gamma_l} N_{\hat{\gamma}_l}$ . Moreover,  $\dot{\gamma}_l^{-1} N_{\hat{\gamma}_l} \dot{\gamma}_l = N_{\hat{\gamma}_l}$ . Thus, if we take  $x'$  to be the element  $n^{\hat{\gamma}_l} \in N_{\hat{\gamma}_l}$  in the decomposition

$$(m')^{-1} a_{w_1}(n')^{-1} n_l a_{w_1}(n') = n^{\hat{\gamma}_l} n^{\hat{\gamma}_l} \in N_{\gamma_l} N_{\hat{\gamma}_l}, \quad (11)$$

then

$$m' x' = a_{w_1}(n')^{-1} n_l a_{w_1}(n') (n^{\hat{\gamma}_l})^{-1},$$

and

$$\begin{aligned} n \dot{\gamma}_1 \cdots \dot{\gamma}_l &= (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_{l-1} \circ \dot{\gamma}_{l-1}) n_l a_{w_1}(n') (n^{\hat{\gamma}_l})^{-1} \dot{\gamma}_l \\ &= (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_{l-1} \circ \dot{\gamma}_{l-1}) n_l \dot{\gamma}_l (\dot{\gamma}_l^{-1} a_{w_1}(n') \dot{\gamma}_l) (\dot{\gamma}_l^{-1} (n^{\hat{\gamma}_l})^{-1} \dot{\gamma}_l). \end{aligned}$$

Set

$$m'' = \dot{\gamma}_l^{-1} (n^{\hat{\gamma}_l})^{-1} \dot{\gamma}_l \in N_{\hat{\gamma}}$$

and let

$$n_l \dot{\gamma}_l = (n_l \circ \dot{\gamma}_l) a_l m_l$$

be the Iwasawa decomposition for  $n_l \dot{\gamma}_l$ , so  $a_l \in A$  and  $m_l \in N_{\gamma_l}$ . Then

$$n \dot{\gamma}_1 \cdots \dot{\gamma}_l = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_l \circ \dot{\gamma}_l) a_l m_l (\dot{\gamma}_l^{-1} a_{w_1}(n') \dot{\gamma}_l) m'',$$

or

$$n \dot{\gamma}_1 \cdots \dot{\gamma}_l = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_l \circ \dot{\gamma}_l) a_l (\dot{\gamma}_l^{-1} a_{w_1}(n') \dot{\gamma}_l) m''', \quad (12)$$

where

$$m''' = (\dot{\gamma}_l^{-1} a_{w_1}(n') \dot{\gamma}_l)^{-1} m_l (\dot{\gamma}_l^{-1} a_{w_1}(n') \dot{\gamma}_l) m'' \in N_{\gamma_l} N_{\hat{\gamma}_l} = N.$$

Therefore, with this choice of  $x' \in N_{\gamma_l}$ , the element  $n = n'x = n' \dot{w}_1 x' \dot{w}_1^{-1} \in N_w$  satisfies (9). Notice that since  $N_{\gamma_l}$  normalizes  $N_{\hat{\gamma}_l}$ , we can first decompose  $(m')^{-1} \in N$  with respect to the decomposition  $N = N_{\gamma_l} N_{\hat{\gamma}_l}$  to get

$$(m')^{-1} = m^{\gamma_l} m^{\hat{\gamma}_l} \quad (13)$$

with  $m^{\gamma_l} \in N_{\gamma_l}$  and  $m^{\hat{\gamma}_l} \in N_{\hat{\gamma}_l}$ . Then

$$x' = n^{\hat{\gamma}_l} = m^{\gamma_l} a_{w_1}(n')^{-1} n_l a_{w_1}(n') \in N_{\gamma_l}. \quad (14)$$

To summerize, we set

$$f : N_{w_1} \times N_{\gamma_l} \longrightarrow N_w : (n', n_l) \longmapsto n' \dot{w}_1 (m^{\gamma_l} a_{w_1}(n')^{-1} n_l a_{w_1}(n')) \dot{w}_1^{-1} \in N_w, \quad (15)$$

and define

$$F_w(n_1, \dots, n_l) = n = n' \dot{w}_1 (m^{\gamma_l} a_{w_1}(n')^{-1} n_l a_{w_1}(n')) \dot{w}_1^{-1} \in N_w. \quad (16)$$

Then  $F_w : N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l} \rightarrow N_w$  is a well-defined map and

$$F_w = f \circ (F_{w_1} \times id).$$

It is also clear now that  $F_w$  is a diffeomorphism.

**Q.E.D.**

Formula (12) also gives the following recursive formula for  $a_w(n)$ :

$$a_w(n) = a_l \dot{\gamma}_l^{-1} a_{w_1}(n') \dot{\gamma}_l, \quad (17)$$

where  $n_l \dot{\gamma}_l = (n_l \circ \dot{\gamma}_l) a_l m_l$  is the Iwasawa decomposition for  $n_l \dot{\gamma}_l$ . Now for each  $j = 1, \dots, l$ , let

$$n_j \dot{\gamma}_j = (n_j \circ \dot{\gamma}_j) a_j m_j$$

be the Iwasawa decomposition. We get from (17) that

$$a_w(n) = a_l(\dot{\gamma}_l^{-1} a_{l-1} \dot{\gamma}_l)(\dot{\gamma}_l^{-1} \dot{\gamma}_{l-1}^{-1} a_{l-2} \dot{\gamma}_{l-1} \dot{\gamma}_l) \cdots (\dot{\gamma}_l^{-1} \dot{\gamma}_{l-1}^{-1} \cdots \dot{\gamma}_2^{-1} a_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{l-1} \dot{\gamma}_l), \quad (18)$$

or, since  $A$  is commutative,

$$\dot{w} a_w(n) \dot{w}^{-1} = \prod_{j=1}^l \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_{j-1}} \sigma_{\gamma_j}(a_j).$$

We know that in the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  coordinates,  $a_j$  is given by

$$a_j = \exp\left(\frac{1}{2}(1 + |z_j|^2)\check{H}_{\gamma_j}\right)$$

for  $j = 1, \dots, l$ . Thus

$$\begin{aligned} \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_{j-1}} \sigma_{\gamma_j}(a_j) &= \sigma_{\gamma_2} \cdots \sigma_{\gamma_{j-1}} \exp\left(-\frac{1}{2} \log(1 + |z_j|^2)\check{H}_{\gamma_j}\right) \\ &= \exp\left(-\frac{1}{2} \log(1 + |z_j|^2)\check{H}_{\alpha_j}\right), \end{aligned}$$

where, for each  $j = 1, \dots, l$ ,

$$\alpha_j = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_{j-1}}(\gamma_j). \quad (19)$$

Recall that

$$\{\alpha_1, \alpha_2, \dots, \alpha_l\} = R_w^+ = \{\alpha > 0 : w^{-1}\alpha < 0\}.$$

Thus we have

$$\dot{w} a_w(n) \dot{w}^{-1} = \prod_{j=1}^l \exp\left(-\frac{1}{2} \log(1 + |z_j|^2)\check{H}_{\alpha_j}\right). \quad (20)$$

Let

$$\beta_j = -w^{-1}\alpha_j = -\sigma_l \sigma_{l-1} \cdots \sigma_j(\gamma_j) = \sigma_l \sigma_{l-1} \cdots \sigma_{j+1}(\gamma_j), \quad (21)$$

i.e.,

$$\begin{aligned} \beta_1 &= \sigma_{\gamma_l} \sigma_{\gamma_{l-1}} \cdots \sigma_{\gamma_2}(\gamma_1) \\ \beta_2 &= \sigma_{\gamma_l} \sigma_{\gamma_{l-1}} \cdots \sigma_{\gamma_3}(\gamma_2) \\ &\dots \\ \beta_{l-1} &= \sigma_{\gamma_l}(\gamma_{l-1}) \\ \beta_l &= \dot{\gamma}_l. \end{aligned}$$

We then know that

$$\{\beta_1, \beta_2, \dots, \beta_l\} = R_{w^{-1}}^+ = \{\beta > 0 : w\beta < 0\}.$$

We also know that  $\ll \beta_j, \beta_j \gg = \ll \alpha_j, \alpha_j \gg = \ll \gamma_j, \gamma_j \gg$  for each  $j = 1, \dots, l$ . This fact will be used later.

The following theorem now follows immediately from (20).

**Theorem 2.5** *In the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$ -coordinates, the function  $a_w$  on  $N_w$  defined by (4) is explicitly given by*

$$a_w(n) = \prod_{j=1}^l \exp\left(\frac{1}{2} \log(1 + |z_j|^2) \check{H}_{\beta_j}\right), \quad (22)$$

where the  $\beta_j$ 's are given by (21).

**Remark 2.6** This formula for  $a_w(n)$  also follows from a product formula found by Doug Pickrell in [P].

We now look at the left invariant Haar measure on  $N_w$ .

**Theorem 2.7** *In the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$ -coordinates, a left invariant (and thus also bi-invariant) Haar measure on  $N_w$  is given as*

$$dn = \lambda_w \prod_{j=1}^l (1 + |z_j|^2)^{\frac{2 \ll \rho, \beta_j \gg}{\ll \beta_j, \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j, \quad (23)$$

where

$$\lambda_w = \frac{1}{(-2\pi i)^l} \prod_{j=1}^l \frac{2 \ll \rho, \beta_j \gg}{\ll \beta_j, \beta_j \gg} = \prod_{j=1}^l \frac{i \ll \rho, \beta_j \gg}{\pi \ll \beta_j, \beta_j \gg} \quad (24)$$

is such that

$$\int_{N_w} a_w(n)^{-4\rho} dn = 1.$$

**Proof.** Again we prove by induction on  $l(w)$ . When  $l(w) = 1$  so  $w = \sigma_\gamma$  for a simple root  $\gamma$ , we have

$$dn_\gamma = -\frac{1}{2\pi i} dz_1 \wedge d\bar{z}_1.$$

Now for  $w$  with  $l(w) = l > 1$ , we use the same notation as that in the proof of Theorem 2.1. Let

$$N_{\alpha_l} = \dot{w}_1 N_{\gamma_l} \dot{w}_1^{-1}$$

be the subgroup of  $N$  with Lie algebra  $\mathbb{C}E_{\alpha_l}$ , where  $\alpha_l = w_1(\gamma_l)$ . Then the multiplication map

$$\mu : N_{w_1} \times N_{\alpha_l} \longrightarrow N : (n', n_{\alpha_l}) \longmapsto n' n_{\alpha_l}$$

is a diffeomorphism. Since  $N$  is unipotent, we have, under the map  $\mu$ ,

$$dn = \lambda dn' dn_{\alpha_l},$$

where  $\lambda$  is a constant to be determined later, and we take

$$dn_{\alpha_l} = du \wedge d\bar{u}$$

if  $N_{\alpha_l}$  is parametrized by  $\{u, \bar{u}\}$  via  $n_{\alpha_l} = \exp(uE_{\alpha_l})$ .

Consider now the parametrization of  $N_w$  by  $\mathbb{C}^l$  via  $F_w$ . Write  $n = F_w(z_1, \bar{z}_1, \dots, z_l, \bar{z}_l)$  if  $n = F_w(n_1, \dots, n_l)$  where  $n_j = \exp(z_j \check{E}_{\gamma_j})$  for each  $j$ . Let again

$$n' = F_{w_1}(z_1, \bar{z}_1, \dots, z_{l-1}, \bar{z}_{l-1}) \in N_{w_1}.$$

Recall that the element  $m' \in N$  is given in (10) and that  $m^{\gamma_l} \in N_{\gamma_l}$  is given in (13). Write

$$m^{\gamma_l} = \exp(m(z_1, \bar{z}_1, \dots, z_{l-1}, \bar{z}_{l-1}) \check{E}_{\gamma_l}).$$

Then we know from (16) that

$$n = n' \exp((m(z_1, \bar{z}_1, \dots, z_{l-1}, \bar{z}_{l-1}) + a_{w_1}(n')^{-\gamma_l} z_l) v_l E_{\alpha_l}),$$

where  $v_l \in \mathbb{C}$  is such that  $Ad_{\dot{w}_1} \check{E}_{\gamma_l} = v_l E_{\alpha_l}$ . Set

$$u = v_l(m(z_1, \bar{z}_1, \dots, z_{l-1}, \bar{z}_{l-1}) + a_{w_1}(n')^{-\gamma_l} z_l).$$

Assume that  $dn'$  is given as in the theorem for  $w_1$ . This means, noting the definition of the  $\beta_j$ 's, that

$$\begin{aligned} dn' &= \lambda_{w_1} \prod_{j=1}^{l-1} (1 + |z_j|^2)^{\frac{2 \ll \rho, \sigma_{\gamma_l} \beta_j \gg}{\ll \sigma_{\gamma_l} \beta_j, \sigma_{\gamma_l} \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j \\ &= \lambda_{w_1} \prod_{j=1}^{l-1} (1 + |z_j|^2)^{-\frac{2 \ll \gamma_l, \beta_j \gg}{\ll \beta_j, \beta_j \gg}} (1 + |z_j|^2)^{\frac{2 \ll \rho, \beta_j \gg}{\ll \beta_j, \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j. \end{aligned}$$

Here we have just used the fact that  $\rho - \sigma_{\gamma_l} \rho = \gamma_l$ . On the other hand, by Theorem 2.5, we have

$$a_{w_1}(n') = \prod_{j=1}^{l-1} \exp\left(\frac{1}{2} \log(1 + |z_j|^2) \check{H}_{\sigma_{\gamma_l}(\beta_j)}\right),$$

so

$$a_{w_1}(n')^{-2\gamma_l} = \prod_{j=1}^{l-1} (1 + |z_j|^2)^{\frac{\ll -2\gamma_l, \sigma_{\gamma_l}\beta_j \gg}{\ll \sigma_{\gamma_l}\beta_j, \sigma_{\gamma_l}\beta_j \gg}} = \prod_{j=1}^{l-1} (1 + |z_j|^2)^{\frac{2 \ll \gamma_l, \beta_j \gg}{\ll \beta_j, \beta_j \gg}}.$$

Therefore,

$$\begin{aligned} dn &= \lambda dn' dn_{\alpha_l} = \lambda' a_{w_1}(n')^{-2\gamma_l} dn' (dz_l \wedge d\bar{z}_l) \\ &= \lambda' \lambda_{w_1} \left( \prod_{j=1}^{l-1} (1 + |z_j|^2)^{\frac{2 \ll \rho, \beta_j \gg}{\ll \beta_j, \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j \right) \wedge (dz_l \wedge d\bar{z}_l), \end{aligned}$$

where  $\lambda' = \lambda|v_l|^2$  is a new constant to be determined later. Since  $\beta_l = \gamma_l$  is a simple root, we have

$$\frac{2 \ll \rho, \gamma_l \gg}{\ll \gamma_l, \gamma_l \gg} = 1.$$

Thus

$$dn = \lambda' \lambda_{w_1} \prod_{j=1}^l (1 + |z_j|^2)^{\frac{2 \ll \rho, \beta_j \gg}{\ll \beta_j, \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j.$$

Using Theorem 2.5, we see that the integral

$$\int_N a_w(n)^{-4\rho} dn$$

is now a product of 1-dimensional ones and is easily calculated. The constant  $\lambda_w = \lambda' \lambda_{w_1}$  must be given by (24) for the above integral to be equal to 1.

**Q.E.D.**

**Example 2.8** We recall that for the complex group  $G$  considered as a real Lie group, the  $c$ -**function** for a Weyl group element  $w$  is defined to be (see [H], Chapter IV, §6)

$$c_w(\lambda) = \int_{\bar{N}_w} a(\bar{n})^{-(i\lambda+2\rho)} d\bar{n},$$

where  $\bar{N}_w \subset N_-$  is the “opposite” of  $N_w$  and  $a(\bar{n})$  is the  $A$ -component in the Iwasawa decomposition of  $\bar{n} \in \bar{N}_w$ . In our notation, we have

$$c_{w^{-1}}(\lambda) = \int_{N_w} a_w(n)^{-(i\lambda+2\rho)} dn.$$

Using our formulas for  $a_w(n)$  and for  $dn$  in the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  coordinates, one immediately reduces the integral to a product of 1-dimensional ones and gets

$$\begin{aligned}
c_{w^{-1}}(\lambda) &= \lambda_w \prod_{j=1}^{l(w)} \int_{\mathbb{R}^2} (1 + |z_j|^2)^{-\frac{\ll i\lambda + 2\rho, \beta_j \gg}{\ll \beta_j, \beta_j \gg} + \frac{2 \ll \rho, \beta_j \gg}{\ll \beta_j, \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j \\
&= \lambda_w \prod_{j=1}^{l(w)} \int_{\mathbb{R}^2} (1 + x_j^2 + y_j^2)^{-\frac{\ll i\lambda, \beta_j \gg}{\ll \beta_j, \beta_j \gg} - 1} (-2i) dx_j dy_j \\
&= \lambda_w \prod_{j=1}^{l(w)} (-2\pi i) \int_0^\infty (1 + r_j^2)^{-\frac{\ll i\lambda, \beta_j \gg}{\ll \beta_j, \beta_j \gg} - 1} 2r_j dr_j \\
&= (-2\pi i)^{l(w)} \lambda_w \prod_{j=1}^{l(w)} \frac{\ll \beta_j, \beta_j \gg}{\ll i\lambda, \beta_j \gg} \quad \text{if } \operatorname{Re} \ll i\lambda, \beta_j \gg > 0 \text{ for each } j \\
&= \prod_{j=1}^{l(w)} \frac{\ll 2\rho, \beta_j \gg}{\ll i\lambda, \beta_j \gg} \quad \text{if } \operatorname{Re} \ll i\lambda, \beta_j \gg > 0 \text{ for each } j.
\end{aligned}$$

This is the well-known formula of Harish-Chandra (see Theorem 5.7 in [H]). As we have mentioned in the Introduction, our calculation here is easier because we have pushed the induction argument that is normally used in the calculations for the  $c$ -functions into the calculations for  $a_w(n)$  and  $dn$ .

**Example 2.9** Consider the example of  $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ . We take  $w$  be to the longest Weyl group element  $w_0 = (1, 2)(2, 3)(1, 2)$ . In this case, parametrize  $N_{w_0} = N$  by complex coordinates

$$(u_1, u_2, u_3) \mapsto n = \begin{pmatrix} 1 & u_1 & u_3 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\dot{\gamma}_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \dot{\gamma}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \quad \dot{\gamma}_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so

$$\dot{w} = \dot{\gamma}_1 \dot{\gamma}_2 \dot{\gamma}_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The Iwasawa decomposition of  $n\dot{w}$  in  $SL(3, \mathbb{C})$  is

$$n\dot{w} = \begin{pmatrix} -u_3 & -u_1 & -1 \\ -u_2 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{u_3}{\Delta_3} & -\frac{u_1(1+|u_2|^2) - \bar{u}_2 u_3}{\Delta_2 \Delta_3} & -\frac{1}{\Delta_2} \\ -\frac{u_2}{\Delta_3} & -\frac{1+|u_2|^2 - u_1 u_2 \bar{u}_3}{\Delta_2 \Delta_3} & \frac{\bar{u}_1}{\Delta_2} \\ -\frac{1}{\Delta_3} & \frac{\bar{u}_2 + u_1 \bar{u}_3}{\Delta_2 \Delta_3} & \frac{\bar{u}_3 - \bar{u}_1 \bar{u}_2}{\Delta_2} \end{pmatrix} \begin{pmatrix} \Delta_3 & \frac{\bar{u}_2 + u_1 \bar{u}_3}{\Delta_3} & \frac{\bar{u}_3}{\Delta_3} \\ 0 & \frac{\Delta_2}{\Delta_3} & \frac{\bar{u}_1(1+|u_2|^2) - u_2 \bar{u}_3}{\Delta_2 \Delta_3} \\ 0 & 0 & \frac{1}{\Delta_2} \end{pmatrix}$$

where

$$\Delta_1 = \sqrt{1+|u_1|^2}, \quad \Delta_2 = \sqrt{1+|u_1|^2+|u_1 u_2 - u_3|^2}, \quad \Delta_3 = \sqrt{1+|u_2|^2+|u_3|^2}.$$

On the other hand, for  $z_1, z_2$  and  $z_3$  in  $\mathbb{C}$ , let

$$n_1 = \begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad n_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \quad n_3 = \begin{pmatrix} 1 & z_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} & (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2)(n_3 \circ \dot{\gamma}_3) \\ &= \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \begin{pmatrix} i z_1 & i & 0 \\ i & -i \bar{z}_1 & 0 \\ 0 & 0 & \varepsilon_1 \end{pmatrix} \begin{pmatrix} \varepsilon_2 & 0 & 0 \\ 0 & i z_2 & i \\ 0 & i & -i \bar{z}_2 \end{pmatrix} \begin{pmatrix} i z_3 & i & 0 \\ i & -i \bar{z}_3 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \\ &= \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \begin{pmatrix} -\varepsilon_2 z_1 z_3 - i z_2 & -\varepsilon_2 z_1 + i z_2 \bar{z}_3 & -\varepsilon_3 \\ -\varepsilon_2 z_3 + i \bar{z}_1 z_2 & -\varepsilon_2 - i \bar{z}_1 z_2 \bar{z}_3 & \bar{z}_1 \varepsilon_3 \\ -\varepsilon_1 & \varepsilon_1 \bar{z}_3 & -i \varepsilon_1 \varepsilon_3 \bar{z}_2 \end{pmatrix}, \end{aligned}$$

where  $\varepsilon_j = \sqrt{1+|z_j|^2}$  for  $j = 1, 2, 3$ . By setting

$$n \circ \dot{w} = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2)(n_3 \circ \dot{\gamma}_3),$$

we get the following coordinate change between our coordinates  $\{(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3)\}$  and the  $u$ 's:

$$u_1 = z_1, \quad u_2 = \frac{\varepsilon_2 z_3 - i \bar{z}_1 z_2}{\varepsilon_1}, \quad u_3 = \frac{\varepsilon_2 z_1 z_3 + i z_2}{\varepsilon_1},$$

or

$$z_1 = u_1, \quad z_2 = i \frac{u_1 u_2 - u_3}{\Delta_1}, \quad z_3 = \frac{\bar{u}_1 u_3 + u_2}{\Delta_2}.$$

Notice that this is not a holomorphic change of coordinates. Thus the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3\}$  coordinates are not complex. Now in the  $u$ -coordinates, we have

$$a_w(n) = \begin{pmatrix} \Delta_3 & 0 & 0 \\ 0 & \frac{\Delta_2}{\Delta_3} & 0 \\ 0 & 0 & \frac{1}{\Delta_2} \end{pmatrix}.$$

Under the coordinate change, we have

$$\Delta_1 = \varepsilon_1, \quad \Delta_2 = \varepsilon_1 \varepsilon_2, \quad \Delta_3 = \varepsilon_2 \varepsilon_3.$$

Thus we get  $a_w(n)$  in the  $\{(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3)\}$  coordinates as:

$$a_w(n) = \begin{pmatrix} \varepsilon_2 \varepsilon_3 & 0 & 0 \\ 0 & \frac{\varepsilon_1}{\varepsilon_3} & 0 \\ 0 & 0 & \frac{1}{\varepsilon_1 \varepsilon_2} \end{pmatrix}.$$

This is the same as what one would obtain from Theorem 2.5. Similarly, the left invariant Haar measure  $dn$  is, up to a constant multiple, given in the  $u$ -coordinates by

$$dn = du_1 \wedge d\bar{u}_1 \wedge du_2 \wedge d\bar{u}_2 \wedge du_3 \wedge d\bar{u}_3.$$

After the change of coordinates, we get

$$dn = \varepsilon_2^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3.$$

Again, this is the same as what one would obtain from Theorem 2.7.

The following proposition will be used in Section 4.

**Proposition 2.10** *We have*

$$F_w(e, \dots, e, n_j, e, \dots, e) = (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1}) n_j (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1})^{-1} \in N_{\alpha_j}$$

for  $j = 1, \dots, l$  and  $n_j \in N_j$ , and

$$(F_w)_*(0) \left( \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1} \wedge \cdots \wedge \frac{\partial}{\partial z_l} \wedge \frac{\partial}{\partial \bar{z}_l} \right) = \left( \prod_{j=1}^l \frac{i}{\ll \alpha_j, \alpha_j \gg} \right) E_{\alpha_1} \wedge iE_{\alpha_1} \wedge \cdots \wedge E_{\alpha_l} \wedge iE_{\alpha_l},$$

where  $(F_w)_*(0)$  is the differential of  $F_w$  at 0.

**Proof.** Set  $n = (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1}) n_j (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1})^{-1}$ . Let

$$n_j \dot{\gamma}_j = (n_j \circ \dot{\gamma}_j) a_j m_j$$

be the Iwasawa decomposition of  $n_j \dot{\gamma}_j$ , where  $m_j \in N_{\gamma_j}$ . Then

$$n_j \dot{\gamma}_j \dot{\gamma}_{j+1} \cdots \dot{\gamma}_l = (n_j \circ \dot{\gamma}_j) (\dot{\gamma}_{j+1} \cdots \dot{\gamma}_l) (\dot{\gamma}_{j+1} \cdots \dot{\gamma}_l)^{-1} a_j (\dot{\gamma}_{j+1} \cdots \dot{\gamma}_l) (\dot{\gamma}_{j+1} \cdots \dot{\gamma}_l)^{-1} m_j (\dot{\gamma}_{j+1} \cdots \dot{\gamma}_l)$$

Set

$$\begin{aligned} a'_j &= (\dot{\gamma}_{j+1} \cdots \dot{\gamma}_l)^{-1} a_j (\dot{\gamma}_{j+1} \cdots \dot{\gamma}_l) \in A \\ m'_j &= (\dot{\gamma}_{j+1} \cdots \dot{\gamma}_l)^{-1} m_j (\dot{\gamma}_{j+1} \cdots \dot{\gamma}_l) \in N. \end{aligned}$$

Then

$$n \dot{\gamma}_1 \cdots \dot{\gamma}_l = \dot{\gamma}_1 \cdots \dot{\gamma}_{j-1} (n_j \circ \dot{\gamma}_j) (\dot{\gamma}_{j+1} \cdots \dot{\gamma}_l) a'_j m'_j.$$

Thus

$$n \circ (\dot{\gamma}_1 \cdots \dot{\gamma}_l) = \dot{\gamma}_1 \cdots \dot{\gamma}_{j-1} (n_j \circ \dot{\gamma}_j) \dot{\gamma}_{j+1} \cdots \dot{\gamma}_l$$

Hence

$$F_w(e, \dots, e, n_j, e, \dots, e) = n.$$

Write  $n_j = \exp(z_j \check{E}_{\gamma_j})$ . Then

$$\begin{aligned} (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1}) n_j (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1})^{-1} &= \exp(z_j \text{Ad}_{\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1}}(\check{E}_{\gamma_j})) \\ &= \exp\left(\sqrt{\frac{2}{\ll \gamma_j, \gamma_j \gg}} z_j \text{Ad}_{\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1}}(E_{\gamma_j})\right). \end{aligned}$$

Since  $\sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_{j-1}}(\gamma_j) = \alpha_j$  by definition, we have

$$\text{Ad}_{\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1}}(E_{\gamma_j}) = c_j E_{\alpha_j}$$

for some complex number  $c_j$ . Write  $z_j = x_j + iy_j$  and  $c_j = u_j + iv_j$ . Using the fact that  $\ll \gamma_j, \gamma_j \gg = \ll \alpha_j, \alpha_j \gg$ , we get

$$\begin{aligned} (F_w)_*(0) \left( \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial \bar{z}_j} \right) &= \frac{i}{2} (F_w)_*(0) \left( \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial y_j} \right) \\ &= \frac{i}{\ll \alpha_j, \alpha_j \gg} (u_j E_{\alpha_j} + v_j (i E_{\alpha_j})) \wedge (-v_j E_{\alpha_j} + u_j (i E_{\alpha_j})) \\ &= \frac{i}{\ll \alpha_j, \alpha_j \gg} |c_j|^2 E_{\alpha_j} \wedge i E_{\alpha_j}. \end{aligned}$$

Since both root vectors  $E_{\gamma_j}$  and  $E_{\alpha_j}$  have length 1 with respect to the  $K$ -invariant Hermitian form on  $\mathfrak{g}$  induced by  $\mathfrak{k}$ , we have  $|c_j|^2 = 1$ . The statement about  $(F_w)_*(0)$  now follows immediately from this.

**Q.E.D.**

We now look at the  $T$ -action on  $N_w$  by conjugations:

$$T \times N_w \longrightarrow N_w : (t, n) \longmapsto t n t^{-1}.$$

For a given  $t \in T$ , set

$$\begin{aligned} t_1 &= t \\ t_2 &= \dot{\gamma}_1 t \dot{\gamma}_1^{-1} \\ t_3 &= \dot{\gamma}_2 \dot{\gamma}_1 t (\dot{\gamma}_2 \dot{\gamma}_1)^{-1} \\ &\dots \\ t_l &= (\dot{\gamma}_{l-1} \dot{\gamma}_{l-2} \dots \dot{\gamma}_1) t (\dot{\gamma}_{l-1} \dot{\gamma}_{l-2} \dots \dot{\gamma}_1)^{-1}. \end{aligned}$$

Equip  $N_{\gamma_1} \times N_{\gamma_2} \times \dots \times N_{\gamma_l}$  with the  $T$ -action given by

$$t \cdot (n_1, n_2, \dots, n_l) = (t_1 n_1 t_1^{-1}, t_2 n_2 t_2^{-1}, \dots, t_l n_l t_l^{-1}). \quad (25)$$

**Proposition 2.11** 1) With respect to the  $T$ -actions on  $N_w$  by conjugation and on  $N_{\gamma_1} \times N_{\gamma_2} \times \dots \times N_{\gamma_l}$  as given by (25), the map  $F_w$  is  $T$ -equivariant;

2) In the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  coordinates, the  $T$ -action on  $N_w$  is given by

$$t \cdot (z_1, \bar{z}_1, \dots, z_l, \bar{z}_l) = (t^{\alpha_1} z_1, t^{-\alpha_1} \bar{z}_1, \dots, t^{\alpha_l} z_l, t^{-\alpha_l} \bar{z}_l), \quad (26)$$

where, recall,  $\alpha_j = \sigma_{\gamma_1} \sigma_{\gamma_2} \dots \sigma_{\gamma_{j-1}}(\gamma_j)$  for  $1 \leq j \leq l$ .

**Proof.** For each  $j = 1, 2, \dots, l$ , we have

$$(t_j n_j t_j^{-1}) \circ \dot{\gamma}_j = t_j (n_j \circ \dot{\gamma}_j) (\dot{\gamma}_j t_j^{-1} \dot{\gamma}_j^{-1}) = t_j (n_j \circ \dot{\gamma}_j) t_{j+1}^{-1},$$

where  $t_{l+1}$  is defined to be equal to  $(\dot{\gamma}_l \dot{\gamma}_{l-1} \dots \dot{\gamma}_1) t (\dot{\gamma}_l \dot{\gamma}_{l-1} \dots \dot{\gamma}_1)^{-1} = \dot{w}^{-1} t \dot{w}$ . Thus,

$$\prod_{j=1}^l (t_j n_j t_j^{-1}) \circ \dot{\gamma}_j = t \left( \prod_{j=1}^l (n_j \circ \dot{\gamma}_j) \right) \dot{w}^{-1} t \dot{w}.$$

On the other hand, let  $n = F_w(n_1, \dots, n_l) \in N_w$ . We have

$$(t n t^{-1}) \circ \dot{w} = t (n \circ \dot{w}) (\dot{w}^{-1} t \dot{w}).$$

It now follows from the definition of  $F_w$  that

$$F_w(t_1 n_1 t_1^{-1}, \dots, t_l n_l t_l^{-1}) = t n t^{-1}.$$

Write  $t = \exp(iH) \in T$  for  $H \in \mathfrak{a}$  and  $n_j = \exp(z_j \check{E}_{\gamma_j}) \in N_{\gamma_j}$ . Then (26) follows from the following calculation:

$$\begin{aligned} t_j n_j t_j^{-1} &= \exp(z_j \operatorname{Ad}_{t_j} \check{E}_{\gamma_j}) \\ &= \exp(z_j e^{i\sigma_{\gamma_1} \sigma_{\gamma_2} \dots \sigma_{\gamma_{j-1}}(\gamma_j)(H)} \check{E}_{\gamma_j}) \\ &= \exp(t^{\alpha_j} z_j \check{E}_{\gamma_j}). \end{aligned}$$

**Q.E.D.**

### 3 The Bruhat-Poisson structure

Recall that a Poisson structure on a manifold  $M$  ([W1]) is a bivector field  $\pi$  on  $M$  such that the bracket operation on the algebra  $C^\infty(M)$  of smooth functions on  $M$  defined by

$$\{\phi, \varphi\} = \pi(d\phi, d\varphi), \quad \phi, \varphi \in C^\infty(M)$$

satisfies Jacobi's identity. The condition on  $\pi$  is that

$$[\pi, \pi] = 0,$$

where  $[\ , \ ]$  denotes the Schouten bracket on the space of multivector fields on  $M$  ([Ko]).

The bivector field  $\pi$  can also be regarded as the bundle map

$$\tilde{\pi} : T^*M \longrightarrow TM : (\tilde{\pi}(\alpha), \beta) = \pi(\alpha, \beta).$$

When  $\pi$  is of maximal rank ( $M$  is then necessarily even dimensional), the bundle map  $\tilde{\pi}$  is invertible, and the 2-form  $\omega$  on  $M$  defined by  $\omega(x, y) = \pi(\tilde{\pi}^{-1}(x), \tilde{\pi}^{-1}(y))$  is closed and non-degenerate and is thus a symplectic 2-form. In general, the image of  $\pi$  defines a (generally singular) involutive distribution on  $M$ . It has integrable submanifolds which inherit symplectic structures [W1]. They are called the symplectic leaves of  $\pi$  in  $P$ . Therefore, symplectic manifolds are special cases of Poisson manifolds and every Poisson manifold is a disjoint union of symplectic manifolds.

The Bruhat-Poisson structure on  $K/T$  comes from a Poisson structure  $\pi$  on  $K$  defined by

$$\pi = \Lambda^r - \Lambda^l,$$

where

$$\Lambda = \frac{1}{2} \sum_{\alpha > 0} X_\alpha \wedge Y_\alpha \in \mathfrak{k} \wedge \mathfrak{k}$$

with  $X_\alpha$  and  $Y_\alpha$  given by (1), and  $\Lambda^r$  (resp.  $\Lambda^l$ ) is the right (resp. left) invariant bi-vector field on  $K$  with value  $\Lambda$  at the identity element  $e$ . We summarize some properties of  $\pi$  in the next theorem. For details, see [STS] [S] and [L-W].

**Theorem 3.1** *a) The bi-vector field  $\pi$  defines a Poisson structure on  $K$ ;*

*b) Equip  $K \times K$  with the product Poisson structure  $\pi \oplus \pi$ . Then the multiplication map*

$$K \times K \longrightarrow K : (k_1, k_2) \longmapsto k_1 k_2$$

*is a Poisson map, making  $(K, \pi)$  into a Poisson Lie group;*

*c) The symplectic leaves of  $\pi$  in  $K$  are precisely the orbits of  $AN$  in  $K$  for the action given by (2). These are the Bruhat cells in  $K$ : for each Weyl group element  $w \in W$  with a fixed representative  $\dot{w}$  of  $w$  in  $K$ , the symplectic leaf through  $\dot{w}$  is the  $N_w$ -orbit  $C_{\dot{w}} = N_w \circ \dot{w}$  introduced in Section 2. For each  $t \in T$ , the subset  $C_{\dot{w}} t$  is the symplectic leaf through the point  $\dot{w} t$ . As  $w$  runs over  $W$  and  $t$  over  $T$ , these are all the symplectic leaves in  $K$ ;*

*d) Both left and right translations by elements in  $T$  leave  $\pi$  invariant;*

*e) The image of  $\pi$  under the projection map  $K \rightarrow K/T$  is a well defined bi-vector field on  $K/T$  which we still denote by  $\pi$ . It defines a Poisson structure on  $K/T$  called the Bruhat-Poisson structure;*

*f) Symplectic leaves of the Bruhat-Poisson structure on  $K/T$  are precisely the Bruhat cells  $\Sigma_w$ , for  $w \in W$ , in  $K/T$  (and thus the name);*

*g) With respect to the left translations by elements in  $K$ , the Bruhat-Poisson structure is  $T$ -invariant but not  $K$ -invariant; The action map*

$$K \times K/T \longrightarrow K/T : (k_1, k_2/T) \longmapsto k_1 k_2/T$$

*is a Poisson map, making  $(K/T, \pi)$  into a Poisson homogeneous  $(K, \pi)$ -space.*

Therefore, for each  $w \in W$  with a fixed representative  $\dot{w}$  in  $K$ , both  $C_{\dot{w}}$  and  $\Sigma_w$  inherit symplectic structures as symplectic leaves in  $K$  and  $K/T$  respectively, and the projection from  $C_{\dot{w}}$  to  $\Sigma_w$  is a symplectic diffeomorphism. Moreover, the symplectic structure on  $\Sigma_w$  is invariant under the action of  $T$  by left translations. The goal of this section is to write down both the

symplectic structure - thus also the Liouville measure - on  $\Sigma_w$  and the moment map for the  $T$ -action in the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  coordinates. Here, we regard  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  as coordinates on  $\Sigma_w$  via the parametrization of  $\Sigma_w$  by  $N_w$ ,

Property b) of  $\pi$  on  $K$  is called the multiplicativity of  $\pi$ . As we will see shortly, it is exactly this property that enables us to decompose, as symplectic manifolds, the higher dimensional  $C_{\dot{w}}$ 's into products of 2-dimensional ones. In fact, this is also the motivation for introducing the coordinates  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$ .

As in Section 2, let  $l = l(w)$  and fix a reduced decomposition

$$w = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_l}.$$

For each  $j = 1, \dots, l$ , let  $\dot{\gamma}_j$  be a representative of  $\sigma_{\gamma_j}$  in  $K$ , so  $\dot{w} = \dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_l$  is a representative of  $w$  in  $K$ . Property c) of  $\pi$  on  $K$  says that the symplectic leaf through  $\dot{\gamma}_j$  is the 2-dimensional cell  $C_{\dot{\gamma}_j}$ . Theorem 2.1 (see Remark 2.2) says that the map

$$C_{\dot{\gamma}_1} \times C_{\dot{\gamma}_1} \times \cdots \times C_{\dot{\gamma}_l} \longrightarrow C_{\dot{w}}. \quad (27)$$

is a diffeomorphism.

**Proposition 3.2** *The map in (27) is a symplectic diffeomorphism, where the left hand side has the product symplectic structure.*

**Proof.** (Notice how the multiplicativity of  $\pi$  is used in the proof.) The inclusion map

$$C_{\dot{\gamma}_1} \times C_{\dot{\gamma}_1} \times \cdots \times C_{\dot{\gamma}_l} \hookrightarrow K \times K \times \cdots \times K \quad (l\text{-copies})$$

is a Poisson map because each  $C_{\dot{\gamma}_j}$  is a symplectic leaf and thus a Poisson submanifold of  $K$ . The multiplicativity of  $\pi$  says that the multiplication map

$$K \times K \times \cdots \times K \longrightarrow K : (k_1, k_2, \dots, k_l) \longmapsto k_1 k_2 \cdots k_l$$

is a Poisson map. Thus, composing the two, we see that

$$C_{\dot{\gamma}_1} \times C_{\dot{\gamma}_1} \times \cdots \times C_{\dot{\gamma}_l} \longrightarrow K : (k_1, k_2, \dots, k_l) \longmapsto k_1 k_2 \cdots k_l$$

is a Poisson map. But it has its image in  $C_{\dot{w}}$  which is a Poisson submanifold of  $K$ . Thus, regarded as a map to  $C_{\dot{w}}$ , the above is a Poisson map, and thus a Poisson and therefore a symplectic diffeomorphism.

**Q.E.D.**

It now remains to determine the symplectic structure on the 2-dimensional leaves. Recall that for each simple root  $\gamma$ , we have a Lie group homomorphism

$$\Phi_\gamma : SL(2, \mathbb{C}) \longrightarrow G$$

which maps  $SU(2)$  to  $K$ .

**Proposition 3.3** (See also [S]) *For each simple root  $\gamma$ , equip  $SU(2)$  with the Poisson structure given by*

$$\pi_\gamma = \Lambda_\gamma^r - \Lambda_\gamma^l,$$

where

$$\Lambda_\gamma = \frac{1}{4} \ll \gamma, \gamma \gg \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in su(2) \wedge su(2),$$

and  $\Lambda_\gamma^r$  (resp.  $\Lambda_\gamma^l$ ) denotes the right (resp. left) invariant bi-vector field on  $SU(2)$  with value  $\Lambda_\gamma$  at the identity element. Then 1)  $\Psi_\gamma : (SU(2), \pi_\gamma) \rightarrow (K, \pi)$  is a Poisson map, and 2) the symplectic leaf of  $\pi_\gamma$  in  $SU(2)$  through the point  $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$  is

$$C_0 = \left\{ \begin{pmatrix} \frac{iz}{\sqrt{1+|z|^2}} & \frac{i}{\sqrt{1+|z|^2}} \\ \frac{i}{\sqrt{1+|z|^2}} & \frac{-i\bar{z}}{\sqrt{1+|z|^2}} \end{pmatrix} : z \in \mathbb{C} \right\}.$$

Using  $(z, \bar{z})$  as coordinates on  $C_0$ , the induced symplectic structure is given by

$$\Omega = \frac{i}{\ll \gamma, \gamma \gg} \frac{1}{1+|z|^2} dz \wedge d\bar{z}.$$

**Proof.** 1) Think of  $\mathfrak{a} + \mathfrak{n}$  as a real Lie algebra and identify  $\mathfrak{a} + \mathfrak{n}$  with  $\mathfrak{k}^*$  using the imaginary part of the Killing form as the pairing. The fact that  $\Psi_\gamma$  is a Poisson map then follows from the fact that the subspace  $\Psi_\gamma(su(2))^\perp$  of  $\mathfrak{a} + \mathfrak{n}$  consisting of all elements that annihilate  $\Psi_\gamma(su(2))$  is an ideal of  $\mathfrak{a} + \mathfrak{n}$ . See [L-W].

2) As a special case of Theorem 3.1 (up to a constant multiple), the symplectic leaves of  $\pi_\gamma$  in  $SU(2)$  are either 2 or 0-dimensional. We know from Section 2 that the set  $C_0$  is the symplectic leaf through the point  $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ .

Write an element of  $SU(2)$  as

$$u = \begin{pmatrix} \xi & \eta \\ -\bar{\eta} & \bar{\xi} \end{pmatrix}.$$

The Poisson brackets defined by  $\pi_\gamma$  are

$$\begin{aligned}\{\xi, \bar{\xi}\} &= -i \ll \gamma, \gamma \gg |\eta|^2 & \{\xi, \eta\} &= \frac{1}{2}i \ll \gamma, \gamma \gg \xi\eta \\ \{\xi, \bar{\eta}\} &= \frac{1}{2}i \ll \gamma, \gamma \gg \xi\bar{\eta} & \{\eta, \bar{\eta}\} &= 0.\end{aligned}$$

Set

$$\xi = \frac{iz}{\sqrt{1+|z|^2}}, \quad \eta = \frac{i}{\sqrt{1+|z|^2}},$$

so

$$z = \xi/\eta, \quad \bar{z} = \bar{\xi}/\bar{\eta}.$$

Using the Leibniz rule for the Poisson bracket, we get

$$\begin{aligned}\{z, \bar{z}\} &= \{\xi/\eta, \bar{\xi}/\bar{\eta}\} \\ &= \frac{1}{|\eta|^2} \{\xi, \bar{\xi}\} - \frac{\bar{\xi}}{|\eta|^2 \bar{\eta}} \{\xi, \eta\} - \frac{\xi}{|\eta|^2 \eta} \{\eta, \bar{\eta}\} \\ &= -i \ll \gamma, \gamma \gg (1 + \frac{|\xi|^2}{|\eta|^2}) \\ &= -i \ll \gamma, \gamma \gg (1 + |z|^2).\end{aligned}$$

Thus the symplectic 2-form  $\Omega$  on  $C_0$  is given as stated.

**Q.E.D.**

**Theorem 3.4** *In the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  coordinates, the induced symplectic structure on  $\Sigma_w$  (as a symplectic leaf of the Bruhat-Poisson structure) and the Liouville volume form  $\mu_w$  associated to  $\Omega_w$  are given by*

$$\Omega_w = \sum_{j=1}^l \frac{i}{\ll \alpha_j, \alpha_j \gg} \frac{1}{1+|z_j|^2} dz_j \wedge d\bar{z}_j \quad (28)$$

$$\mu_w := \frac{1}{l!} (\Omega_w)^l = \prod_{j=1}^l \frac{i}{\ll \alpha_j, \alpha_j \gg} \frac{1}{1+|z_j|^2} dz_j \wedge d\bar{z}_j. \quad (29)$$

The moment map for the  $T$ -action on  $\Sigma_w$  by left translations satisfying  $\phi_w(0) = 0$  is given by

$$\phi_w : \Sigma_w \longrightarrow \mathfrak{t}^* \cong \mathfrak{a} : \phi_w = \sum_{j=1}^l \left(-\frac{1}{2} \log(1+|z_j|^2) \check{H}_{\alpha_j}\right). \quad (30)$$

Here we are identifying  $\mathfrak{t}^*$  with  $\mathfrak{a}$  by using the imaginary part of the Killing form as the pairing.

**Proof.** The formula for  $\Omega_w$  follows immediately from Propositions 3.2 and 3.3 and the fact that  $\ll \gamma_j, \gamma_j \gg = \ll \alpha_j, \alpha_j \gg$ .

The formula for the moment  $\phi_w$  follows from the explicit formula for the  $T$ -action on  $\Sigma_w$  in the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  coordinates as given in Proposition 2.11: let  $iH \in \mathfrak{t}$ , where  $H \in \mathfrak{a}$ . By Proposition 2.11, the generating vector field of the  $T$  action on  $\Sigma_w$  in the direction of  $iH$  is given by

$$V_{iH} = \sum_{j=1}^l \alpha_j(H) \left( -y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j} \right).$$

Set

$$V_{iH} \lrcorner \Omega_w = d \langle \phi_w, iH \rangle.$$

We see that the only solution of  $\phi_w$  that satisfies this equation for all  $H \in \mathfrak{a}$  and the condition that  $\phi_w(0) = 0$  is given by (30).

**Q.E.D.**

The following two corollaries follow immediately by comparing the formulas in Theorems 2.5, 2.7 and 3.4 (see also identity (20)).

**Corollary 3.5** *Think of the moment map  $\phi_w$  as a map from  $N_w$  to  $\mathfrak{a}$  via the parametrization of  $\Sigma_w$  by  $N_w$ . Then*

$$\phi_w = Ad_{\dot{w}} \log a_w(n).$$

**Corollary 3.6** *Think of the Haar measure  $dn$  of  $N_w$  as a volume form on  $\Sigma_w$  via the parametrization of  $\Sigma_w$  by  $N_w$ . It is related to the Liouville measure  $\mu_w$  by*

$$\mu_w = \left( \prod_{j=1}^l \frac{\pi}{\ll \rho, \beta_j \gg} \right) a_w(n)^{-2\rho} dn.$$

We have thus connected the moment map  $\phi_w$  and the Liouville measure  $\mu_w$  with the familiar map  $a_w : N_w \rightarrow A$  and the Haar measure  $dn$  on  $N_w$ . Such a connection is desirable for understanding the geometry of the Bruhat-Poisson structure. We have arrived at this by comparing their formulas in coordinates. This is why we wanted to write down the formulas for  $a_w$  and  $dn$  in the  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  coordinates in Section 2.

## 4 Kostant's theorem on $H(G/B)$

In [K], Kostant constructs, for each element  $w$  in the Weyl group  $W$ , an explicit  $K$ -invariant closed differential form  $s^w$  on  $X$  with  $\deg(s^w) = 2l(w)$ , such that the cohomology classes of the  $s^w$ 's form a basis of  $H(X, \mathbb{C})$  that, up to scalar multiples, is dual to the basis of the homology of  $X$  formed by the closures of the Bruhat cells in  $X$ . We now recall the definition of these forms in more details.

Let  $C = \oplus C^\bullet$  be the space of all  $K$ -invariant differential forms on  $G/B \cong K/T$ . By identifying  $\mathfrak{g}$  and  $\mathfrak{g}^*$  via the Killing form  $\ll, \gg$  so that

$$(\mathfrak{g}/\mathfrak{h})^* \cong \mathfrak{n}_- + \mathfrak{n},$$

we can identify

$$C^\bullet \cong \wedge^\bullet(\mathfrak{n}_- + \mathfrak{n})^T.$$

Introduce the operators  $E$  and  $L_0$  on  $C^\bullet$  by

$$E = 2 \sum_{\alpha > 0} ad_{E_{-\alpha}} \otimes ad_{E_\alpha}$$

and

$$= \begin{cases} L_0(E_{-\alpha_1} \wedge E_{\alpha_2} \wedge \cdots \wedge E_{-\alpha_p} \otimes E_{\beta_1} \wedge E_{\beta_2} \wedge \cdots \wedge E_{\beta_q}) & \text{if } \|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2 = 0 \\ 0 & \text{if } \|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2 = 0 \\ \frac{1}{\|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2} & \text{if } \|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2 \neq 0 \end{cases}$$

Set

$$R = -L_0 E.$$

It is a nilpotent operator.

Now let  $w \in W$  be a Weyl group element. As before, use  $l$  to denote  $l(w)$ . Let

$$\{\alpha_1, \alpha_2, \dots, \alpha_l\} = \{\alpha > 0 : w^{-1}\alpha < 0\},$$

and let

$$\beta_j = -w^{-1}\alpha_j$$

so that

$$\{\beta_1, \beta_2, \dots, \beta_l\} = \{\beta > 0 : w\beta < 0\}.$$

Set

$$h^{w^{-1}} = \left(\frac{i}{2}\right)^l E_{-\beta_1} \wedge E_{-\beta_2} \wedge \cdots \wedge E_{-\beta_l} \otimes E_{\beta_1} \wedge E_{\beta_2} \wedge \cdots \wedge E_{\beta_l},$$

and

$$s^w = (1 - R)^{-1} h^{w^{-1}} = h^{w^{-1}} + Rh^{w^{-1}} + R^2 h^{w^{-1}} + \cdots. \quad (31)$$

**Theorem 4.1 (Kostant [K])** 1). *The forms  $s^w$  for  $w \in W$  are closed, and their cohomology classes form a basis of the de Rham cohomology of  $G/B \cong K/T$  (with complex coefficients);*

2). *Let  $j_w : N_w \rightarrow \Sigma_w : n \mapsto nw/T$  be the parametrization map. Then*

$$j_{w_1}^*(s^w|_{\Sigma_{w_1}}) = \begin{cases} 0 & \text{if } l(w_1) = l(w) \text{ but } w_1 \neq w \\ a_w(n)^{-2(\rho - w^{-1}\rho)}(dn)_1 & \text{if } w_1 = w, \end{cases}$$

where  $s^w|_{\Sigma_{w_1}} = i_{w_1}^* s^w$  if  $i_{w_1}$  is the inclusion map of  $\Sigma_{w_1}$  into  $K/T$ , the map  $a_w : N_w \rightarrow A$  is, as before, defined by (4), and  $(dn)_1$  is the left invariant Haar measure on  $N_w$  normalized by the condition

$$((dn)_1, E_{\alpha_1} \wedge iE_{\alpha_1} \wedge E_{\alpha_2} \wedge iE_{\alpha_2} \wedge \cdots \wedge E_{\alpha_l} \wedge iE_{\alpha_l}) = 1.$$

3)

$$\int_{\Sigma_w} s^w = \prod_{j=1}^l \frac{\pi}{\ll \rho, \alpha_j \gg}.$$

The purpose of this section is to relate the form  $s^w$  with the Liouville volume form  $\mu_w$  on  $\Sigma_w$  induced by the Bruhat-Poisson structure. This is now easy due to Corollaries 3.5 and 3.6. We will also give a simple proof of 3).

We first need to relate the left Haar measures  $(dn)_1$  and  $dn$  as given in Theorem 2.7.

**Lemma 4.2** *We have*

$$(dn)_1 = \prod_{j=1}^l \frac{\pi}{\ll \rho, \beta_j \gg} dn.$$

**Proof.** This follows from Proposition 2.10 and the fact that  $\ll \alpha_j, \alpha_j \gg = \ll \beta_j, \beta_j \gg$  for each  $j = 1, \dots, l$ .

**Q.E.D.**

**Theorem 4.3** When restricted to the Schubert cell  $\Sigma_w$ , Kostant's harmonic form  $s^w$  is related to the Liouville volume form  $\mu_w$  on  $\Sigma$  by

$$s^w|_{\Sigma_w} = (a_w)^{2w^{-1}\rho} \mu_w = e^{\langle \phi_w, 2iH_\rho \rangle} \mu_w. \quad (32)$$

**Proof.** This is a direct consequence of Corollaries 3.5 and 3.6. Explicitly, we have

$$s^w|_{\Sigma_w} = \prod_{j=1}^l \frac{i}{\ll \alpha_j, \alpha_j \gg} (1 + |z_j|^2)^{-\frac{2 \ll \rho, \alpha_j \gg}{\ll \alpha_j, \alpha_j \gg} - 1} dz_j \wedge d\bar{z}_j \quad (33)$$

$$\langle \phi_w, 2iH_\rho \rangle = - \sum_{j=1}^l \frac{2 \ll \rho, \alpha_j \gg}{\ll \alpha_j, \alpha_j \gg} \log(1 + |z_j|^2) \quad (34)$$

and

$$\mu_w = \prod_{j=1}^l \frac{i}{\ll \alpha_j, \alpha_j \gg} (1 + |z_j|^2)^{-1} dz_j \wedge d\bar{z}_j.$$

Thus we have (32).

**Q.E.D.**

From the explicit formula for  $s^w|_{\Sigma_w}$ , we immediately get

$$\begin{aligned} \int_{\Sigma_w} s^w &= \prod_{j=1}^l \int_{\mathbb{R}^2} \frac{i}{\ll \alpha_j, \alpha_j \gg} (1 + |z_j|^2)^{-\frac{2 \ll \rho, \alpha_j \gg}{\ll \alpha_j, \alpha_j \gg} - 1} dz_j \wedge d\bar{z}_j \\ &= \prod_{j=1}^l \frac{\pi}{\ll \rho, \alpha_j \gg}. \end{aligned}$$

This integral was first calculated in [K-K] using induction on  $l(w)$ . Again, as in the case of the  $c$ -functions, our simple proof is due to the fact that the induction argument has been pushed to the calculations for  $a_w(n)$  and  $dn$ . This is in fact a special case of the  $c$ -function with  $i\lambda = -2w^{-1}\rho$ .

**Remark 4.4** When  $w = w_0$  is the longest Weyl group element, the form  $s^{w_0}$  is a  $K$ -invariant volume form so it coincides with the Haar measure on  $K/T$ . When restricted to the biggest cell  $\Sigma_{w_0}$ , the Liouville volume form  $\mu_{w_0}$ , the form  $s^{w_0}$ , and the Haar measure  $dn$  for  $N_{w_0} = N$  are related by

$$s^{w_0}|_{\Sigma_{w_0}} = (a_{w_0})^{-2\rho} \mu_{w_0} = \left( \prod_{\alpha > 0} \frac{\pi}{\ll \rho, \alpha \gg} \right) a_{w_0}(n)^{-4\rho} dn.$$

**Remark 4.5** The function  $\langle \phi_w, 2iH_\rho \rangle$  on  $\Sigma_w$  is the Hamiltonian function for the generating vector field  $\theta_0$  of the  $T$ -action in the direction of  $2iH_\rho$ . This vector field is intrinsic to the Bruhat-Poisson structure in the sense that it is its modular vector field. As  $|z_j| \rightarrow +\infty$ , the function  $\langle \phi_w, 2iH_\rho \rangle \rightarrow -\infty$ . Thus the modular vector field  $\theta_0$  is not globally Hamiltonian on  $K/T$ . We say that the Bruhat-Poisson structure is not unimodular. See [W2] [B-Z] [E-L-W].

**Remark 4.6** The second identity in (32) expresses the form  $s^w|_{\Sigma_w}$  totally in terms of data coming from the Bruhat-Poisson structure. In particular, it says that the integral  $\int_{\Sigma_w} s^w$  is of the Duistermaat-Heckman type. Wanting to see this was another motivation for this work. Theorem 4.3 can be used to describe generators of the so-called Poisson - de Rham cohomology of the Bruhat-Poisson structure. We do this in [E-L].

## 5 Appendix: Relation to the Bott-Samelson coordinates

As before, let  $w \in W$  be a Weyl group element with a fixed reduced decomposition:

$$w = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_l}$$

where  $l = l(w)$ . Then we have the Lie group homomorphism  $\Psi_{\gamma_j} : SL(2, \mathbb{C}) \rightarrow G$  and the element

$$\dot{\gamma}_j = \Psi_{\gamma_j} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in K$$

for each  $j = 1, \dots, l$ . Set again  $\dot{w} = \dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_l \in K$ . Then the map

$$\begin{aligned} F'_w : N_{\gamma_1} \times \cdots \times N_{\gamma_l} &\longrightarrow N_w : \\ (n_1, n_2, \dots, n_l) &\longmapsto n_1 \dot{\gamma}_1 n_2 \dot{\gamma}_2 \cdots n_l \dot{\gamma}_l \dot{w}^{-1} \\ &= n_1 (\dot{\gamma}_1 n_2 \dot{\gamma}_1^{-1}) \cdots (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{l-1}) n_l (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{l-1})^{-1} \end{aligned}$$

is a holomorphic diffeomorphism. Thus, if we parametrize  $N_{\gamma_j}$  by  $\mathbb{C}$  via

$$z'_j \longmapsto n(z'_j) := \Psi_{\gamma_j} \begin{pmatrix} 1 & z'_j \\ 0 & 1 \end{pmatrix} = \exp(z'_j \check{E}_{\gamma_j}) \in N_{\gamma_j}$$

for each  $j$ , we get a parametrization of  $N_w$  by  $\mathbb{C}^l$ :

$$\mathbb{C}^l \longrightarrow N_w : (z'_1, z'_2, \dots, z'_l) \longmapsto F'_w(n(z'_1), n(z'_2), \dots, n(z'_l)).$$

We call it the Bott-Samelson parametrization of  $N_w$  and call  $\{z'_1, z'_2, \dots, z'_l\}$  the **Bott-Samelson coordinates** on  $N_w$  because of the close relation to the Bott-Samelson desingularization of the Schubert variety  $X_w$  [J].

The difference between our coordinates  $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_l, \bar{z}_l\}$  and the Bott-Samelson coordinates  $\{z'_1, z'_2, \dots, z'_l\}$  can be described as follows: by composing  $F_w$  (in Theorem 2.1) and  $F'_w$  with the map

$$j_w : N_w \longrightarrow \Sigma_w : n \longmapsto n\dot{w}/B,$$

we can think of both  $F_w$  and  $F'_w$  as mapping  $N_{\gamma_1} \times \dots \times N_{\gamma_l}$  diffeomorphically to  $\Sigma_w$ . The map  $F'_w$  first sends  $(n_1, n_2, \dots, n_l)$  to the product of  $n_1\dot{\gamma}_1 n_2\dot{\gamma}_2 \dots n_l\dot{\gamma}_l$  in  $G$  and then projects the product to  $G/B$ . But for  $F_w$ , we first pick up the  $K$ -component  $k_j$  in the Iwasawa decomposition of  $n_j\dot{\gamma}_j$  for each  $j$ , multiply the  $k_j$ 's inside  $K$  and then project the product to  $K/T \cong G/B$ .

Consider now the change of coordinates

$$(z'_1, z'_2, \dots, z'_l) = I_w(z_1, z_2, \dots, z_l).$$

We can get a recursive formula for  $I_w$  from the proof of Theorem 2.7. Clearly, when  $w$  is a simple reflection, the map  $I_w$  is the identity map. For a general  $w$  with the reduced decomposition  $w = \sigma_{\gamma_1} \sigma_{\gamma_2} \dots \sigma_{\gamma_l}$ , let again  $w_1 = \sigma_{\gamma_1} \sigma_{\gamma_2} \dots \sigma_{\gamma_{l-1}}$  so that  $w = w_1 \sigma_{\gamma_l}$ . Keeping the same notation as in the proof of Theorem 2.7, we know that  $I_w$  is given by

$$\begin{cases} (z'_1, z'_2, \dots, z'_{l-1}) = I_{w_1}(z_1, z_2, \dots, z_{l-1}) \\ z'_l = m(z_1, z_2, \dots, z_{l-1}) + a_{w_1}(n)^{-\gamma_l} z_l, \end{cases}$$

where, recall,  $m(z_1, z_2, \dots, z_{l-1}) \in \mathbb{C}$  is such that  $\exp(m(z_1, z_2, \dots, z_{l-1})\check{E}_{\gamma_l})$  is the  $N_{\gamma_l}$ -component of  $(m')^{-1} \in N$  with respect to the decomposition  $N = N_{\gamma_l} N_{\check{\gamma}_l}$ , and  $m' \in N$  is the  $N$ -component in the Iwasawa decomposition of  $n\dot{w}_1$ . The change of coordinates is in general not complex because the function  $m(z_1, z_2, \dots, z_{l-1})$  is in general not holomorphic. This is also seen from the following example.

**Example 5.1** For  $G = SL(3, \mathbb{C})$  and  $w = (1, 2)(2, 3)(1, 2)$ , the Bott-Samelson parametrization of  $N_w$  is by

$$(z'_1, z'_2, z'_3) \longmapsto \begin{pmatrix} 1 & z'_1 & z'_1 z'_3 + i z'_2 \\ 0 & 1 & z'_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

The change of coordinates between  $(z_1, z_2, z_3)$  and  $\{z'_1, z'_2, z'_3\}$  are (see Example 2.9)

$$z'_1 = z_1, \quad z'_2 = \varepsilon_1 z_2 \quad z'_3 = \frac{\varepsilon_2 z_3 - i \bar{z}_1 z_2}{\varepsilon_1},$$

or

$$z_1 = z'_1, \quad z_2 = \frac{z'_2}{\eta_1} \quad z_3 = \frac{\eta_1^2 z'_3 + i \bar{z}'_1 z'_2}{\eta_2},$$

where  $\varepsilon_j = \sqrt{1 + |z_j|^2}$  for  $j = 1, 2$  and  $\eta_1 = \sqrt{1 + |z'_1|^2}$ ,  $\eta_2 = \sqrt{1 + |z'_1|^2 + |z'_2|^2}$ .

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