on $-\infty < t < \infty$ of

$$\dot{x}_m = -a_m x_m + b_m s(t)$$

$$\dot{x}_p = -a_p x_p + b_p[\theta x_p + ks(t)]$$

$$\dot{\hat{a}}_p = \dot{\hat{b}}_p = \dot{\theta} = \dot{k} = 0$$

and $e_i \equiv e_c \equiv e_k \equiv e_\theta \equiv 0$ so that

$$0 = b_p(\theta - \theta^*)x_p + b_p(k - k^*)s(t) \tag{23}$$

and

$$0 = -(\hat{a}_p - a_p)x_p + (\hat{b}_p - b_p)[\theta x_p + ks(t)]. \tag{24}$$

In particular, $\hat{a}_p$, $\hat{b}_p$, θ, and k are constant on Z_I and $e_c = x_p - x_m \equiv 0$ so that $x_p \equiv x_m$. Thus,

$$\dot{x}_p = -a_m x_p + b_m s(t)$$

and $x_p(t)$ is bounded on $-\infty < t < \infty$. Hence,

$$x_p(t) = \int_{-\infty}^{t} e^{-a_m(t-u)} b_m s(u)\,du. \tag{25}$$

When (25) is used in (23), we can argue as in Section V that the constants $\theta - \theta^*$ and $k - k^*$ must be zero. When (25) is used in (24), we argue the same way that $(\hat{b}_p - b_p)k^* = 0$ and

$$(\hat{a}_p - a_p) + (\hat{b}_p - b_p)\theta^* = 0.$$

Hence, $\hat{b}_p = b_p$ and $\hat{a}_p = a_p$ on Z_I. Thus, for the original system, we see that $e_i(t)$, $e_c(t)$, $e_\theta(t)$, and $e_k(t)$ all tend to zero as $t \to \infty$, while $\theta(t) \to \theta^*$, $k(t) \to k^*$, $\hat{a}_p(t) \to a_p$, and $\hat{b}_p(t) \to b_p$, as desired.

VIII. Concluding Remarks

In the present paper, we developed a new generalized invariance principle (see Sections II and III and Theorems 1–3) to prove that model reference adaptive control does stabilize a given linear system. Our results, which address a scalar adaptive control problem (considered previously in [5]) and an nth-order adaptive control problem (considered previously in [6]), are new (see Sections IV and VI and Theorem 4). In order to demonstrate the wide applicability of our results, we also addressed a class of scalar direct–indirect adaptive control problems considered recently in [8] and [9] (see Section VII).

References

[1] R. K. Miller and G. R. Sell, *Volterra Integral Equations and Topological Dynamics*, Memoirs of Amer. Math. Soc., no. 102, 1970.

[2] ——, "Existence, uniqueness and continuity of solutions of integral equations," *Ann. Math. Pura u. Appl.*, vol. 80, pp. 135–152, 1968.

[3] G. R. Sell, "Compact sets of nonlinear operators," *Funkcialcj Ekvacivj*, vol. 11, pp. 131–138, 1968.

[4] R. K. Miller and A. N. Michel, *Ordinary Differential Equations*. New York: Academic, 1982.

[5] F. M. A. Salam, "On the sufficient richness and parameter convergence in MRAC systems," in *Proc. 1987 IEEE Conf. Decision Contr.*, Los Angeles, CA, Dec. 1987, pp. 13–15.

[6] Y. D. Landau, *Adaptive Control*. New York: Marcel Dekker, 1979.

[7] P. A. Ioannou and P. V. Kokotovic, *Adaptive Systems with Reduced Models*. Berlin–Heidelberg–New York: Springer-Verlag, 1983.

[8] M. A. Duarte and K. S. Narendra, "Combined direct and indirect approach to adaptive control," Cen. Syst. Sci., Yale Univ., New Haven, CT, Tech. Rep. 8711, Sept. 1987.

[9] ——, "Indirect adaptive control using dynamical adjustment of control parameters," Cen. Syst. Sci., Yale Univ., New Haven, CT, Tech. Rep. 8813, Feb. 1988.

[10] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

Convexity of the Largest Singular Value of $e^D M e^{-D}$: A Convexity Lemma

NAM-KIU TSING

Abstract—**A rigorous proof is given for a convexity lemma used by Chu and Doyle to prove the convexity of the largest singular value of $e^D M e^{-D}$ with respect to D on a commuting, convex subset of matrices.**

The structured singular value (SSV), which was originated by Doyle [1], is a useful tool for the analysis and synthesis of feedback systems with structured uncertainties (e.g., see [1], [2]). While the exact value of the SSV is, in general, difficult to compute, one can always bound it from above by the numerical value

$$\inf_{D \in \mathfrak{D}} \bar{\sigma}(e^D M e^{-D}) \tag{1}$$

where $M \in \mathcal{C}^{n \times n}$ is fixed, $\mathfrak{D} \subset \mathcal{R}^{n \times n}$ is a certain (convex) subset of diagonal matrices, and $\bar{\sigma}(M)$ denotes the largest singular value of the matrix M. As the real-valued function $\phi: \mathfrak{D} \to \mathcal{R}$ defined by

$$\phi(D) = \bar{\sigma}(e^D M e^{-D})$$

is convex on $\mathfrak{D}$, the optimization problem (1) is tractable. A rather long proof of the convexity of ϕ can be found in [3]. A much shorter proof was later proposed in [4] (see also [2]), based on the convexity lemma given below. More recently, a short direct proof that does not use the convexity lemma has been given by Sezginer and Overton [5]. The purpose of this note is to prove the convexity lemma which, while intuitive, has apparently not been rigorously proved.

Convexity Lemma [4]: Suppose that $f: \mathcal{R} \to \mathcal{R}$ is continuous, and that for all $s \in \mathcal{R}$, there exists $g_s: \mathcal{R} \to \mathcal{R}$ twice differentiable such that $f(s) = g_s(s)$, $f(t) \geq g_s(t)$ for all t and

$$\left.\frac{d^2}{dt^2} g_s(t)\right|_{t=s} \geq 0.$$

Then f is convex.

Proof of Lemma: Let $f: \mathcal{R} \to \mathcal{R}$ satisfy the hypotheses of the lemma. For any closed interval $[a, b]$ (with $a < b$), let Γ be the graph of f on $[a, b]$, i.e.,

$$\Gamma = \{(t, f(t)) \in \mathcal{R}^2 : t \in [a, b]\}$$

and let K be the convex hull of Γ. Since Γ is compact, K is also compact. Define $\psi: [a, b] \to \mathcal{R}$ by

$$\psi(t) = \max\{u : (t, u) \in K\} \qquad \forall t \in [a, b]$$

(i.e., the graph of ψ is the upper boundary of K). Then ψ is continuous on $[a, b]$. For any $s \in (a, b)$, we consider two cases. First, suppose that $(s, \psi(s))$ is not an extreme point of K. Then ψ is linear on a neighborhood of s. Hence, ψ is infinitely differentiable at s and, in particular, $\psi''(s) = 0$. Second, suppose that $(s, \psi(s))$ is an extreme point of K. Then $f(s) = \psi(s)$. Let $h(t) = c_0 + c_1(t - s)(t \in \mathcal{R})$ be the equation of a supporting line to K at $(s, \psi(s))$. Then

$$c_0 = h(s) = \psi(s) = f(s) = g_s(s) \tag{2}$$

and

$$h(t) \geq \psi(t) \geq f(t) \geq g_s(t) \qquad \forall t \in [a, b]. \tag{3}$$

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Since both h and g_s are differentiable at s, by (2) and (3), we must have

$$g_s'(s) = h'(s) = c_1.$$

Since g_s is twice differentiable at s, we can write

$$g_s(t) = c_0 + c_1(t-s) + c_2(t-s)^2 + \epsilon(t-s)$$

where

$$\lim_{t \to s} \frac{\epsilon(t-s)}{(t-s)^2} = 0.$$

Now, since

$$c_0 + c_1(t-s) = h(t) \geq g_s(t) = c_0 + c_1(t-s) + c_2(t-s)^2 + \epsilon(t-s) \qquad \forall t \in [a, b],$$

we must have $c_2 \leq 0$. However, $c_2 = \frac{1}{2} g_s''(s) \geq 0$ by assumption. Thus, $c_2 = 0$. By (3) again,

$$c_0 + c_1(t-s) = h(t) \geq \psi(t) \geq g_s(t) = c_0 + c_1(t-s) + \epsilon(t-s) \qquad \forall t \in [a, b].$$

Thus,

$$|\psi(t) - c_0 - c_1(t-s)| \leq |\epsilon(t-s)| \qquad \forall t \in [a, b]$$

which shows that ψ is twice differentiable at s and that $\psi''(s) = 0$. Combining the two cases, we see that

$$\psi''(t) = 0 \qquad \forall t \in (a, b).$$

As a result, ψ is linear on $[a, b]$, and thus the graph of f on $[a, b]$ is below the line segment joining $(a, f(a))$ and $(b, f(b))$. Since this is true for any $a < b$, f is convex. □

To conclude this note, and for the sake of ease of reference, we now reproduce the proof of the convexity of ϕ given in [4]. Following [5], the proof does not make use of the specific definition of $\mathfrak{D}$, but only relies on the fact that $\mathfrak{D}$ is a commuting convex set (i.e., $X, Y \in \mathfrak{D}$ and $0 < t < 1$ will imply $XY = YX$ and $tX + (1-t)Y \in \mathfrak{D}$).

Proof of Convexity of ϕ [4]: Since $\mathfrak{D}$ is commuting, if $X, Y \in \mathfrak{D}$ and $0 \leq t \leq 1$, then

$$e^{tX+(1-t)Y} M e^{-(tX+(1-t)Y)} = e^{t(X-Y)} (e^Y M e^{-Y}) e^{-t(X-Y)}.$$

Thus, it suffices to prove convexity of $f(t) = \bar{\sigma}(e^{tD} M e^{-tD})$ over $\mathfrak{R}$ for arbitrary $D, M \in \mathfrak{C}^{n \times n}$. Define $M_s = e^{sD} M e^{-sD}$ and let u and v be any singular vectors such that

$$f(s) = \bar{\sigma}(M_s) = u^H M_s v.$$

Define

$$g_s(t) = \mathrm{Re}\,(u^H e^{tD} M e^{-tD} v).$$

Since $f(t) \geq g_s(t)$ and

$$\begin{aligned} \left.\frac{d^2}{dt^2} g_s(t)\right|_{t=s} &= \mathrm{Re}\,(u^H (D^2 M_s - 2DM_s D + M_s D^2) v) \\ &= f(s)(u^H D^2 u + v^H D^2 v) - 2\,\mathrm{Re}\,(u^H D M_s D v) \\ &= [u^H D \; v^H D] \begin{bmatrix} f(s)I & -M_s \\ -M_s^* & f(s)I \end{bmatrix} \begin{bmatrix} Du \\ Dv \end{bmatrix} \geq 0 \end{aligned}$$

where * denotes Hermitian conjugation, the results follow from the lemma. □

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References

[1] J. C. Doyle, "Analysis of feedback systems with structured uncertainties," *Proc. IEE*, part D, vol. 129, pp. 242–250, 1982.
[2] A. K. Packard, "What's new with μ: Structured uncertainties in multivariable control," Ph.D. dissertation, Univ. California, Berkeley, 1988.
[3] M. G. Safanov and J. C. Doyle, "Minimizing conservativeness of robustness singular values," in *Multivariable Control*, S. G. Tzafestas, Ed. Amsterdam: Reidel, 1984, pp. 197–207.
[4] J. C. Doyle and C. C. Chu, "Robustness control of multivariable and large scale systems," Final Tech. Rep. for AFOSR, Contract F49620-84-C-0088, Mar. 1986.
[5] R. Sezginer and M. Overton, "The largest singular value of $e^X A_0 e^{-X}$ is convex on convex sets of commuting matrices," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 229–230, Feb. 1990.

Disturbance Rejection and Tracking Using Output Feedforward Control

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Abstract—**The disturbance rejection and tracking problems are known to have particularly nice solutions when full state feedback is used. Although the output feedback problem, in general, has a very difficult solution, it will be seen that for these special cases, the solution involves solving only a linear two-point boundary-value problem, rather than the nonlinear problem associated with the general case.**

Introduction

It has been known for some time that the feedback gains in the disturbance rejection problem do not depend on the disturbance model parameters. In the tracking problem, the feedback gains also do not depend on the model to be tracked. These are two very important control problems, with the attractive property that only the feedforward gains depend on the model for the uncontrollable portion of the system [1]. The optimal solutions do require, however, that the entire substate for the uncontrollable portion be available. If this is not the case, then one must resort to using an observer–estimator or consider using only the measurable states to control the system. The latter choice leads us to a special case of the output feedback problem solved some time ago in the literature [2]. The solution to the general output feedback problem unfortunately involves solving a nonlinear matrix two-point boundary-value problem. The contribution of this brief note is to point out that for two important cases, one can avoid this level of difficulty. The Kronecker algebra is of considerable usefulness in obtaining the solution, which involves a matrix *linear* two-point boundary-value problem. The reader may observe that the solution has applications in the area of large-scale systems where one may consider the disturbance to be the interconnection vector which reflects how the other states of a large system affect the local dynamical model.

Problem Statement, Disturbance Rejection

Consider a state-space model

$$\begin{aligned} \dot{x}_1 &= \hat{A}_{11} x_1 + \hat{A}_{12} x_2 + Bu \\ \dot{x}_2 &= A_{22} x_2 + w \end{aligned} \tag{1}$$

where w is zero-mean white noise with covariance matrix $\hat{Q}$. If both x_1

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