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<td><strong>Author(s)</strong></td>
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A Nonconservative LMI Condition for Stability of Switched Systems With Guaranteed Dwell Time

Graziano Chesi, Senior Member, IEEE, Patrizio Colaneri, Fellow, IEEE, Jose C. Geromel, Member, IEEE, Richard Middleton, Fellow, IEEE, and Robert Shorten, Senior Member, IEEE

Abstract—Ensuring stability of switched linear systems with a guaranteed dwell time is an important problem in control systems. Several methods have been proposed in the literature to address this problem, but unfortunately they provide sufficient conditions only. This technical note proposes the use of homogeneous polynomial Lyapunov functions in the non-restrictive case where all the subsystems are Hurwitz, showing that a sufficient condition can be provided in terms of an LMI feasibility test by exploiting a key representation of polynomials. Several properties are proved for this condition, in particular that it is also necessary for a sufficiently large degree of these functions. As a result, the proposed condition provides a sequence of upper bounds of the minimum dwell time that approximate it arbitrarily well. Some examples illustrate the proposed approach.

Index Terms—Dwell time, homogeneous polynomial, LMI, Lyapunov function, switched system.

I. INTRODUCTION

An important problem in control systems consists of ensuring stability of switched linear systems under a dwell time constraint, see, e.g., [1]–[9]. Several methods have been proposed in the literature for addressing this problem, as in [10], [11] where a condition is provided on the basis of the norm of the transition matrices associated with the system matrices, and as in [12] where a condition is provided by exploiting quadratic Lyapunov functions and LMIs. Unfortunately, these methods provide conditions that are only sufficient.

This technical note addresses this problem by using homogeneous polynomial Lyapunov functions, which have been adopted in the study of uncertain systems [13]–[15], in the non-restrictive case where all the subsystems are Hurwitz. It is shown that a sufficient condition can be provided in terms of an LMI feasibility test by using a representation of polynomials in an extended space and the concept of sum of squares (SOS). Several properties are proved for this condition, in particular that it is also necessary for a sufficiently large degree of the Lyapunov functions. As a result, the proposed condition provides a sequence of upper bounds of the minimum dwell time that approximate...
it arbitrary well. The proposed approach is illustrated by some numerical examples. A preliminary version of this technical note appeared in [16].

Before proceeding it is worth mentioning that SOS techniques have been proposed in the literature for investigating switched systems, as in [17] which addresses stability analysis of switched and hybrid systems under arbitrary switching using polynomial and piecewise polynomial Lyapunov functions. See also the recent survey [18].

The technical note is organized as follows. In Section II the problem is formulated and some preliminary results are given. Section III describes the proposed results. Section IV presents some examples that illustrate the use and benefits of the proposed approach. Lastly, Section V concludes the technical note with some final remarks. The proofs of the proposed results are reported in the Appendix.

II. PRELIMINARIES

The notation used throughout the technical note is as follows: \( \mathbb{N}, \mathbb{R} \) : natural and real number sets; \( \mathbb{R}^n \) : origin of \( \mathbb{R}^n \); \( \mathbb{R}^n \setminus \{0\} \); \( I_n \) : \( n \times n \) identity matrix; \( A^t \) : transpose of \( A \); \( A > 0 \) (\( A \geq 0 \) ) : symmetric positive definite (semidefinite) matrix; \( A \); \( \nabla v(x) \) : first derivative row vector of the function \( v(x) \).

We consider switched linear systems of the form

\[
\dot{x}(t) = A_{\sigma(t)}x(t)
\]

where \( t \geq 0 \), \( x(t) \in \mathbb{R}^n \), and \( \sigma(t) \) is a switching signal taking values in a finite set \( \mathcal{S} = \{1, \ldots, M\} \) (note that the differential equation only holds almost everywhere), and \( A_1, \ldots, A_M \in \mathbb{R}^{n \times n} \) are given matrices. The switching signal is assumed to belong to the set

\[
\mathcal{D}_T = \{ \sigma : \mathbb{R}_+ \rightarrow \mathcal{S} : t_{k+1} - t_k \geq T \}
\]

where \( t_k \) are the switching instants and \( T > 0 \).

Problem. The problem we consider in this technical note is to establish whether (1) is exponentially stable for all \( \sigma(\cdot) \in \mathcal{D}_T \).

Solving this problem allows one to address the minimum dwell time problem, i.e., the computation of the minimum \( T \) ensuring exponential stability of (1) for all \( \sigma(\cdot) \in \mathcal{D}_T \). We define this time as

\[
T_{\text{min}} = \inf\{T : (1) \text{ is exponentially stable for all } \sigma(\cdot) \in \mathcal{D}_T \}.
\]

Let us observe that a necessary condition for (1) to be exponentially stable for all \( \sigma(\cdot) \in \mathcal{D}_T \) is that the matrices \( A_1, \ldots, A_M \) are Hurwitz. Therefore, we assume without loss of generality that \( A_1, \ldots, A_M \) are Hurwitz matrices.

Our starting point for addressing the considered problem is the following result that was given in [12] for guaranteeing an upper bound to the minimum dwell time.

Theorem 1 (see [12]): Assume that, for given \( T > 0 \)

\[
\exists P_i : \left\{ \begin{array}{l}
P_i > 0 \quad \forall \mathcal{S} \\
A_i^TP_i + P_iA_i < 0 \quad \forall \mathcal{S} \\
e^{A_i^TT}P_i e^{A_iT} < P_i \quad \forall \mathcal{S}, i \neq j.
\end{array} \right.
\]

Then, the system is exponentially stable for all \( \sigma(\cdot) \in \mathcal{D}_T \).

The above result deserves a few remarks.

1) For given Hurwitz matrices \( A_1, \ldots, A_M \), there always exist \( T > 0 \) such that (4) holds.

2) The function \( v(x, t) = x^TP_{\sigma(t)}x \) is a piecewise quadratic Lyapunov function for (1) for all \( \sigma(\cdot) \in \mathcal{D}_T \).

III. PROPOSED RESULTS

The idea exploited in this technical note is to adopt homogeneous polynomials. Any homogeneous polynomial \( h(x) \) of degree \( \leq m \) in \( x \in \mathbb{R}^n \) can be expressed as

\[
h(x) = x^m(\text{H} + L_m(\alpha))x^m
\]

where \( x^m(\cdot) \in \mathbb{R}^{m \times m} \) contains all monomials of degree \( m \) in \( x \), where

\[
d(n, m) = \frac{(n + m - 1)!}{(n - 1)!m!}
\]

\( H \in \mathbb{R}^{d(n, m) \times d(n, m)} \) is a symmetric matrix, \( L_m(\alpha) \) is a linear parametrization of the linear subspace

\[
\mathcal{L}_m = \left\{ L = L' : x^m(\cdot)Lx^m = 0 \quad \forall x \in \mathbb{R}^n \right\}
\]

and \( \alpha \in \mathbb{R}^{d(n, m)} \) is a free vector, where

\[
d_{\text{par}}(n, m) = \frac{1}{2}d(n, m)(d(n, m) + 1) - d(n, 2m).
\]

The representation (1) is known as square matrix representation (SMR) and Gram matrix method, and allows one to establish whether a polynomial is SOS via an LMI feasibility test, specifically \( h(x) \) is SOS if and only if there exists \( \alpha \) such that \( H + L_m(\alpha) \geq 0 \). See e.g., [20] and references therein for details on SOS polynomials, and [21] for details on LMIs.

Now, there exists \( \mathcal{A}_{\text{par}} \in \mathbb{R}^{d(n, m) \times d(n, m)} \) satisfying

\[
\frac{d}{dx}x^m = \mathcal{A}_{\text{par}}x^m \quad \forall x \in \mathbb{R}^n.
\]

This matrix, called extended matrix of \( A \), with respect to \( x^m(\cdot) \), can be computed as [15]

\[
\mathcal{A}_{\text{par}} = (K_0^{-1}K_0)^{-1}K_0 \left( \sum_{i=0}^{n-1} L_{m-1-i} \otimes A \otimes I_i \right) K_0
\]

where \( K_0 \) is the matrix satisfying

\[
x^m = K_0x^m(\cdot)
\]

and \( x^{\otimes m} \) denotes the \( m \)-th Kronecker power of \( x \). The following theorem provides a condition for guaranteeing exponential stability of (1) under a dwell time constraint based on homogeneous polynomial Lyapunov functions.
Then, (1) is exponentially stable for all \( \sigma(\cdot) \in D_T \).

Theorem 3 states that one can establish whether there exists a homoge-
neous polynomial Lyapunov functions ensuring that (1) is exponen-
tially stable for all \( \sigma(\cdot) \in D_T \) through an LMI feasibility test.

Let us observe that (13) coincides with (4) for \( m = 1 \), i.e., in the case
where the homogeneous polynomial Lyapunov functions are quadratic.
Let us also observe that, as \( \tilde{T} \) tends to zero, the matrices \( \Pi_i \) tend to a
common matrix \( \Pi \), and (13) tends to the condition provided in \( [15] \) for
robust stability of time-varying polytopic systems based on a common
homogeneous polynomial Lyapunov function.

Table I shows the total number of LMI variables in (13) for some
values of \( n, m \), and \( M \).

The following result provides a monotonicity property of the condi-
tion (13) with respect to \( T \).

Theorem 4: Assume that (13) holds for some \( T = \tilde{T} \) and \( m \). Then,
(13) holds also for \( T = \tilde{T} + \tau \) and \( m \) for all \( \tau \in \mathbb{R}, \tau \geq 0 \).

The following result provides a monotonicity property of the condi-
tion (13) with respect to \( m \).

Theorem 5: Assume that (13) holds for some \( T \) and \( m = \tilde{m} \). Then,(13) also holds for \( T = \tilde{T} \) and \( m = k \tilde{m} \) for all \( k \in \mathbb{N}, k \geq 1 \). We now
give an important result that states that the condition by Theorem 3 is
not only sufficient but also necessary for some \( m \).

Theorem 6: The system (1) is exponentially stable in \( D_T \) if and only
if there exists \( m \) such that (13) holds.

Theorem 6 states that, whenever (1) is exponentially stable in \( D_T \),
there exists a homogeneous polynomial Lyapunov function of bounded
degree that can be found by solving the LMI condition (13). Let us
observe that this result does not contradict the result given in \([22] \)
which states that the degree of a polynomial Lyapunov function is not uni-
formly bounded over the class of asymptotically stable switched linear
system.

Let us indicate with \( T_m \) the smallest upper bound of \( T_{m,\text{min}} \) guaran-
teed by Theorem 3 for a given \( m \), i.e.

\[
T_m = \inf \{ T \in \mathbb{R}_+ \mid T > 0 : (13) \text{ holds} \}. \tag{14}
\]

Due to Theorem 6, one has that the minimum dwell time \( T_{m,\text{min}} \) can be
approximated arbitrary well by the upper bound \( T_m \), i.e.

\[
\forall \varepsilon \in \mathbb{R}_+, \varepsilon > 0, \exists m \in \mathbb{N} \mid m \geq 1 : T_m - T_{m,\text{min}} < \varepsilon. \tag{15}
\]

Moreover, due to Theorem 4, one can calculate \( T_m \) via a bisection
search where at each iteration the condition (13) is tested. This search
is conducted in an interval \([0, \tilde{T}]\) where \( \tilde{T} \) is such that (13) holds for
\( T = \tilde{T} \). Observe that such \( \tilde{T} \) is guaranteed to exist for all \( m \geq 1 \) since
it exists for \( m = 1\) and due to Theorem 5.

IV. EXAMPLES

In this section, examples are presented to illustrate the usefulness of
the proposed approach. The upper bound \( T_m \) provided by condition
(13) is computed by using Matlab and SeDuMi [23] on a personal com-
puter with Windows XP, Pentium 4 3.20 GHz, 2 GB RAM. Each ex-
ample shows the total number of LMI variables in the condition (3) and
the average computational time (ACT) required for testing this condi-
tion in the bisection search used to find \( T_m \). The matrix function \( L_m(\alpha) \)
is computed with the algorithm reported in [24] and available in the
Matlab toolbox SMRROFT [25].

For comparison we consider the upper bound provided in the pio-
neering paper \([10]\), i.e.,

\[
T_{H,M} = \max_i \inf_{\alpha > 0, \beta > 0} \left\{ \alpha : \|A_i^{\alpha}d\| \leq e^{-\alpha \beta T}, \forall \beta > 0 \right\}.
\]

In addition, we consider

\[
T_{L,H} = \inf\left\{ T > 0 \text{ s.t. } \max_i \|\lambda_g \left( \sum_{p=1}^{M} e^{H_p^T} \right) \| < 1, \forall \tilde{T} > 0 \right\}
\]

where \( \lambda_g \) denotes a generic eigenvalue and \( \{B_1, B_2, \ldots, B_M\} \) are
matrices corresponding to any permutation among those of the set
\( \{A_1, A_2, \ldots, A_M\} \). Of course \( T_{H,M} \leq T_{m,\text{min}} \), i.e., \( T_{L,H} \) is a lower
bound of the minimum dwell time.

A. Example 1

Consider for \( n = 2 \) and \( M = 2 \) the matrices

\[
A_1 = \begin{pmatrix} 0 & 1 \\ -10 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -0.1 & -1 \end{pmatrix}.
\]

We have \( T_{H,M} = 2.4132 \) and \( T_{L,H} = 0 \). By using the proposed
approach, we get the following upper bounds:

\[
\begin{array}{c|c|c|c}
m & T_m & \# LMI variables & ACT [s] \\
\hline
1 & 0.1700 & 6 & 0.1297 \\
2 & 0.0451 & 15 & 0.1669 \\
3 & 0.0000 & 29 & 0.1588 \\
4 & 0.0000 & 48 & 0.2068 \\
\end{array}
\]

Since \( T_3 = 0 \), it clearly follows that \( T_{m,\text{min}} = T_3 \) as \( T_3 \) is an upper
bound of \( T_{m,\text{min}} \) and \( T_{m,\text{min}} \) is nonnegative.

In order to illustrate more clearly the use of the condition provided
in Theorem 3, we report hereafter the matrices \( A_{1,m} \) and \( L_m(\alpha) \) involved
in (13) for the case \( m = 2 \) (homogeneous polynomial Lyapunov function
of degree 4):

\[
A_{1,m} = \begin{pmatrix} 0 & 2 \\ -10 & -1 \end{pmatrix}, \quad A_{2,m} = \begin{pmatrix} -0.1 & -4 \\ 0 & -0.2 \end{pmatrix}, \quad L_m(\alpha) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.
\]
B. Example 2

Here we consider an example with $n = 2$ and $M = 3$, specifically

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & -1 \\ 4 & 1 \end{pmatrix}.$$ 

We have $T_{H,M} = 2.3024$ and $T_{L,H} = 0.8006$. With the proposed approach we get the following upper bounds:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$T_m$</th>
<th># LMI variables</th>
<th>ACT [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2989</td>
<td>6</td>
<td>0.1138</td>
</tr>
<tr>
<td>2</td>
<td>1.2439</td>
<td>20</td>
<td>0.2083</td>
</tr>
<tr>
<td>3</td>
<td>1.2427</td>
<td>72</td>
<td>0.2882</td>
</tr>
<tr>
<td>4</td>
<td>1.2427</td>
<td>272</td>
<td>0.4860</td>
</tr>
</tbody>
</table>

The minimum dwell time $T_{\text{min}}$ coincides with $T_3$. This is confirmed by the fact that, taking the periodic signal of period $T = t_2 + t_3$ with $t_2 = t_3 = 1.2427$ as

$$\sigma(t) = \begin{cases} 1, & t \in [kT, kT + t_2), \; k \in \mathbb{N} \\ 2, & t \in [kT + t_2, (k + 1)T), \; k \in \mathbb{N} \end{cases}$$

the associated periodic system $\dot{x}(t) = A_{\sigma(t)}x(t)$ is not asymptotically stable (the maximum modulus of the characteristic multipliers is equal to 1). Hence, $T_{\text{min}} = T_3 = 1.2427$.

C. Example 3

Next, we consider for $n = 3$ and $M = 3$ the matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & -4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 & 1 \\ -2 & -2 & -16 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -5 & -2 & -9 \end{pmatrix}. $$

We have $T_{H,M} = 18.5554$ and $T_{L,H} = 5.0677$. With the proposed approach we get the following upper bounds:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$T_m$</th>
<th># LMI variables</th>
<th>ACT [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11.7934</td>
<td>18</td>
<td>0.1441</td>
</tr>
<tr>
<td>2</td>
<td>11.2439</td>
<td>99</td>
<td>0.5649</td>
</tr>
<tr>
<td>3</td>
<td>11.2439</td>
<td>327</td>
<td>6.1792</td>
</tr>
<tr>
<td>4</td>
<td>11.2439</td>
<td>1386</td>
<td>69.9121</td>
</tr>
</tbody>
</table>

The minimum dwell time $T_{\text{min}}$ coincides with $T_3$. This is confirmed by the fact that, taking the periodic signal of period $T = t_1 + t_2$ with $t_1 = 11.2439$ and $t_2 = 12.2849$ as

$$\sigma(t) = \begin{cases} 1, & t \in [kT, kT + t_1), \; k \in \mathbb{N} \\ 2, & t \in [kT + t_1, (k + 1)T), \; k \in \mathbb{N} \end{cases}$$

the associated periodic system $\dot{x}(t) = A_{\sigma(t)}x(t)$ is not asymptotically stable (the maximum modulus of the characteristic multipliers is equal to 1). Hence, $T_{\text{min}} = T_2 = 11.2439$.

D. Example 4

Lastly, we consider for $n = 2$ and $M = 2$ the example in [26] given by

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \frac{1}{a} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $a$ is a parameter. The problem consists of determining the value $a^*$ for which the system is exponentially stable for all switching signals in $D_0$ (arbitrary switching without dwell time) for all $a \in [1, a^*]$. In [26] it has been shown analytically that $a^* \geq 10$, and that the set of values of $a$ for which there exists a quadratic Lyapunov function is included in the interval $[1, 6]$.

The quantity $a^*$ can be estimated with the proposed approach, which in the case $T = 0$ coincides with the condition provided in [15] for robust stability of time-varying polytopic systems based on a common homogeneous polynomial Lyapunov function. Specifically, a lower bound of $a^*$ can be found through a bisection search where stability over $[1, a]$ is established by using Theorem 3. Let us denote this lower bound as $a^*_m$. The results obtained are as follows:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$a^*_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.8283</td>
</tr>
<tr>
<td>2</td>
<td>9.2911</td>
</tr>
<tr>
<td>3</td>
<td>9.6825</td>
</tr>
<tr>
<td>4</td>
<td>10.4105</td>
</tr>
</tbody>
</table>

V. Conclusion

This technical note has proposed for the first time in the literature a nonconservative LMI condition for ensuring stability of switched linear systems with a guaranteed dwell time. This condition has been derived by exploiting homogeneous polynomial Lyapunov functions and a representation of polynomials in an extended space. As a result, the proposed condition provides a sequence of upper bounds of the minimum dwell time that approximate it arbitrarily well. Future work will investigate the possibility of determining upper bounds of the degree of the homogeneous polynomial Lyapunov functions required to prove stability.

Appendix

Proof of Theorem 3: Suppose that (13) holds, and define

$$v_1(x) = x^{[m]} \Pi_x x^{[m]}.$$ 

The first LMI in (13) clearly implies that $v_1(x)$ is positive definite. Then, we have that

$$\nabla v_1(x)A_i x = x^{[m]} \left( A_i^{[m]} \Pi_i + \Pi_i A_i^{[m]} + L_i^{(ii)} \right) x^{[m]}$$

and hence the second LMI in (13) implies that $\nabla v_i(x)A_i x$ is negative definite. Lastly, it turns out that

$$e^{A_i^{[m]} x} x^{[m]} = e^{A_i^{[m]} x^{[m]}}$$

which implies that

$$v_j(e^{A_i^{[m]} x}) - v_i(x) = x^{[m]} 
\times e^{A_i^{[m]} x^{[m]}} \Pi_j e^{A_i^{[m]} x^{[m]}} - \Pi_j - L_i^{(ii)} x^{[m]}$$

and hence $v_j(e^{A_i^{[m]} x}) - v_i(x)$ is negative definite. Therefore, (1) is exponentially stable for all $\sigma(\cdot) \in D_T$ since (5) holds.
Proof of Theorem 4: Suppose that (13) holds, and define \( v_i(x) = x^{(m)}_r \Pi_r x^{(m)}_r \). Consider any \( \tau \geq 0 \). From the second inequality one has that

\[
v_i(x(\tau)) \leq v_i(x(0))
\]

which implies that

\[
e^{A_i \alpha \Pi_r \alpha} x^{(m)}_r \leq 0.
\]

Pre- and post-multiplying the third inequality of (13) by \( e^{A_i \alpha \Pi_r \alpha} \) and \( e^{A_i \alpha \Pi_r \alpha} \), respectively, one gets that

\[
e^{A_i \alpha \Pi_r \alpha (T + \tau)} x^{(m)}_r e^{A_i \alpha \Pi_r \alpha \tau} < 0.
\]

Lastly, let us observe that

\[
x^{(m)}_r e^{A_i \alpha \Pi_r \alpha \tau} x^{(m)}_r = 0
\]

which implies that

\[
e^{A_i \alpha \Pi_r \alpha} L_m (\alpha, i) x^{(m)}_r \in L_m
\]

or, in other words

\[
\exists \tilde{v}_i, e^{A_i \alpha \Pi_r \alpha} L_m (\alpha, i) e^{A_i \alpha \Pi_r \alpha} \in L_m
\]

Proof of Theorem 5: Let \( \Pi_i, \alpha_i, \Pi_{i,j} \) be such that (13) holds for \( T = m = n_i \), and consider any \( k \in \mathbb{N} \), \( k \geq 1 \). We now show that there exist, \( \Pi_i, \alpha_i, \Pi_{i,j} \) such that (13) holds for \( T = m = n_i \).

Define \( \tilde{v}_i(x) = x^{(n_i)}_r \Pi_r x^{(n_i)}_r \). We have that (5) holds with these functions. Define also the homogeneous polynomial of degree \( 2k n_i \)

\[
\tilde{v}_i(x) = v_i^k(x).
\]

Clearly, (5) holds with \( v_i(x), v_j(x) \) replaced by \( \tilde{v}_i(x), \tilde{v}_j(x) \), respectively. Now, let us define

\[
\Pi_i = K_i^r \Pi_r \Pi_i \Pi_{i,j} K_1
\]

where \( K_i \) is the (full-column) rank matrix satisfying

\[
x^{(n_i)}_r \otimes \alpha_x = K_i x^{(k n_i)}_x.
\]

We have that

\[
\tilde{v}_i(x) = v_i^{(k n_i)}_r \Pi_r x^{(k n_i)}_r \Pi_i \Pi_{i,j} K_1
\]

Then, let us define

\[
\Phi_i = K_2^r \left( \Pi_i^{(k_n)}_r \otimes (\Pi_i^{(k_n)}_r \Pi_i + \Pi_i \Pi_i^{(k n_i)}_r + (\alpha x)_i) \right) K_2
\]

where \( K_2 \) is the (full-column) rank matrix satisfying

\[
x^{(n_i)}_r \otimes \alpha_x = K_2 x^{(k n_i)}_x.
\]

We have that

\[
\nabla \tilde{v}_i(x) = v_i^{(k n_i)}_r \Phi_i x^{(k n_i)}_x \Phi_i < 0
\]

and

\[
\exists \tilde{v}_i, A_i^{(k n_i)} \Pi_i + \Pi_i A_i^{(k n)} + L_{k_n} (\alpha_i) = 7 \Phi_i
\]

because \( A_i^{(k n)} \Pi_i + \Pi_i A_i^{(k n)} \) and \( \Phi_i \) are SMR matrices of the same homogeneous symmetric matrices. Lastly, let us define

\[
\Phi_i, j = K_i^r \left( (e^{A_i \alpha \Pi_r \alpha} \Pi_r e^{A_i \alpha \Pi_r \alpha}) \otimes \Pi_i \otimes \Phi_i \right) K_1
\]

It can be verified that

\[
\tilde{v}_j (e^{A_i \alpha \Pi_r \alpha}) x - \tilde{v}_i (x) = x^{(k n_i)}_x \Phi_i, j \Phi_i, j x^{(k n_i)}_x \Phi_i, j < 0
\]

and

\[
\exists \tilde{v}_j, e^{A_i \alpha \Pi_r \alpha} \Pi_i e^{A_i \alpha \Pi_r \alpha} \Pi_i - L_{k_n} (\alpha_i) = \Phi_i, j
\]

because \( e^{A_i \alpha \Pi_r \alpha} \Pi_i e^{A_i \alpha \Pi_r \alpha} \Pi_i = \Pi_i \) and \( \Phi_i, j \) are SMR matrices of the same homogeneous polynomial.

Proof of Theorem 6: Assume that (1) is exponentially stable for all \( \sigma (\cdot) \in D_T \). From Theorem 2 one has that there exist functions \( v_i(x) \) satisfying (5). As shown in [27], these functions can be chosen of the form \( ||M_i x||^2 \) for some matrix \( M_i \), moreover as discussed in [28] \( ||M_i x||^2 \) converges uniformly on as \( p \to \infty \) and the \( v_i(x) \) can be chosen of the form \( ||M_i x||^2 \) for some finite \( p \). Then, observe that if \( v_i(x) \) satisfies (5) then also \( v_i(x)^n \) satisfies (5) for all \( n \in \mathbb{N} \), \( \alpha 
\]

Define

\[
\mathbf{u}_i(x) = - \nabla \tilde{v}_i(x) A_i x
\]

\[
\mathbf{w}_{i,j}(x) = v_i(x) - v_j (e^{A_i \alpha \Pi_r \alpha} \Pi_i \Pi_j)
\]

The functions \( \mathbf{u}_i(x) \) and \( \mathbf{w}_{i,j}(x) \) are homogeneous polynomials since the \( v_i(x) \) are homogeneous polynomials, and positive definite since (5) holds.

Now, suppose that each \( v_i(x) \) is replaced by \( \bar{v}_i(x) = v_i x^k \) with \( k \in \mathbb{N} \), \( k \geq 1 \). We have that \( u_i(x) \) and \( w_{i,j}(x) \) become

\[
\bar{u}_i(x) = k v_i(x)^{k-1} \mathbf{u}_i(x)
\]

\[
\bar{w}_{i,j}(x) = v_i x^k - v_j (e^{A_i \alpha \Pi_r \alpha} \Pi_i \Pi_j)
\]

Also

\[
\bar{w}_{i,j}(x) = z_{i,j}(x) w_{i,j}(x)
\]

where

\[
z_{i,j}(x) = \sum_{i=0}^{k-1} v_i(x)^{k-1-i}.
\]

For \( \varepsilon \in \mathbb{R} \) define

\[
\bar{u}_{i,j}(x) = k v_i(x)^{k-1} \left( u_i(x) - \varepsilon ||x||^{2p} \right)
\]

\[
\bar{w}_{i,j}(x) = v_i(x)^k - \varepsilon ||x||^{2p}.
\]

Since \( u_i(x) \) and \( w_{i,j}(x) \) are positive definite, it follows that there exists \( \varepsilon > 0 \) such that \( u_i(x) - \varepsilon ||x||^{2p} \) and \( w_{i,j}(x) - \varepsilon ||x||^{2p} \) are positive definite, which implies that \( \bar{u}_{i,j}(x) \) and \( \bar{w}_{i,j}(x) \) are positive definite. Let us observe that, since \( v_i(x) \) is positive definite and SOS, then \( \bar{v}_i(x) \) is positive definite and SOS for all \( k \geq 1 \). Also, since \( u_i(x) - \varepsilon ||x||^{2p} \) is positive definite, from [29] it follows that there exists a sufficiently large \( k \) (denoted as \( k_1 \)) such that \( \bar{u}_{i,j}(x) \) is positive definite and SOS. Similarly, one has that \( w_{i,j}(x) - \varepsilon ||x||^{2p} \) is positive definite and SOS for all \( \varepsilon \in [0, k-1] \) for a sufficiently large \( k \) (denoted as \( k_2 \), which implies that \( \bar{w}_{i,j}(x) \) is positive definite and SOS for all \( k \geq k_2 \).
Summarizing, one has that \( \tilde{r}_i(x), \tilde{u}_i(x) \) and \( \tilde{w}_{i,j}(x) \) are positive definite and SOS for all \( k \geq k_3 \), where \( k_3 = \max \{ k_1, k_2 \} \), for some \( \varepsilon > 0 \).

Then, let us observe that any homogeneous polynomial \( h(x) \) that is SOS can be expressed as in (6) with a positive semidefinite matrix \( P \), see e.g., [20] and references therein. This means that one can write \( \tilde{r}_i(x) = x^T \tilde{P}_i x \) where \( \tilde{P}_i = \tilde{P} + k_3 \tilde{V}_i \geq 0 \). Since \( \tilde{r}_i(x) = \| M_i x \|_{\tilde{P}}^2 \), it is not difficult to see that \( \tilde{V}_i \) can be chosen not only positive semidefinite but also positive definite (just observe that \( M_i \) must have full column rank). Similarly, one can write \( \tilde{u}_i(x) = x^T \tilde{U}_i x \) and \( \tilde{w}_{i,j}(x) = x^T \tilde{W}_{i,j} x \) where \( \tilde{U}_i \geq 0 \) and \( \tilde{W}_{i,j} \geq 0 \). Moreover, one can write \( \tilde{u}_i(x) = x^T \tilde{U}_i x \) and \( \tilde{w}_{i,j}(x) = x^T \tilde{W}_{i,j} x \). Since \( \tilde{u}_i(x) - \tilde{u}_j(x) = -k_{ij}(x)^{-\varepsilon} \| x \|_{\tilde{P}}^{2\varepsilon} \) and \( \tilde{w}_{i,j}(x) - \tilde{w}_{j,i}(x) = -k_{ij}(x)^{-\varepsilon} \| x \|_{\tilde{P}}^{2\varepsilon} \), it is not difficult to see that \( \tilde{V}_i \) and \( \tilde{W}_{i,j} \) can be chosen positive definite (just observe that \( \tilde{V}_i = \tilde{U}_i + \tilde{U}_j \) and \( \tilde{W}_{i,j} = \tilde{W}_{j,i} + \tilde{W}_{i,j} \) with \( \tilde{U}_i > 0 \) and \( \tilde{W}_{i,j} > 0 \)).

Lastly, since \( \Pi_i, -\Pi_i, \Pi = \Pi_i A_i - L_i (\alpha_i) \) and \( \Pi_i + L_i (\alpha_i) - e^{A_i t_0} \Pi e^{A_i t} \) are SMR matrices of \( \tilde{r}_i(x), \tilde{u}_j(x) \) and \( \tilde{w}_{i,j}(x) \), and since the parameterization of the SMR matrices in (13) is complete, it follows that there exist \( \Pi_i, \alpha_i \) and \( \alpha_{i,j} \), such that (13) holds.

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REFERENCES


