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Spot Pricing When Lagrange Multipliers Are Not Unique

Donghan Feng, Member, IEEE, Zhao Xu, Member, IEEE, Jin Zhong, Senior Member, IEEE, and Jacob Østergaard, Senior Member, IEEE

Abstract—Classical spot pricing theory is based on multipliers of the primal problem of an optimal market dispatch, i.e., the solution of the dual problem. However, the dual problem of market dispatch may yield multiple solutions. In these circumstances, spot pricing or any standard pricing practice based on multipliers cannot generate a unique clearing price. Although such situations are rare, they can cause significant uncertainties and complexities in market dispatch. In practice, this situation is solved through simple empirical methods, which may cause additional operations or biased allocation. Based on a strict extension of the principles of spot pricing and surplus allocation, we propose a new pricing methodology that can yield unique, impartial, and robust solution. The new method has been analyzed and compared with other pricing approaches in accordance with spot pricing theory. Case studies support the results of the theoretical analysis, and further demonstrate that the method performs effectively in both uniform-pricing and nodal-pricing markets.

Index Terms—Double-sided auction, duality, nodal price, spot pricing.

NOMENCLATURE

The main notation used throughout the paper is stated next for quick reference. Other symbols are defined as required.

A. Indices and Numbers

- $N_G$: Number of generators in the system.
- $N_D$: Number of loads in the system.
- $N_b$: Number of pricing zones (buses) in the system.
- $N_{mB}$: Number of MDS buses in the system.
- $N_{nB}$: Number of non-MDS buses in the system.
- $N_t$: Number of transmission lines in the system.

B. Variables and Parameters

- $PG_i$: Offer price of generator $i$ ($/MW$).
- $PD_j$: Bid price of load $j$ ($/MW$).
- $G_i$: Generation of generator $i$ (MW).
- $D_j$: Demand of load $j$ (MW).
- $G_{G_i}$: Output upper limit of generator $i$ (MW).
- $G_{G_i}$: Output lower limit of generator $i$ (MW).
- $D_{D_j}$: Maximal demand of load $j$ (MW).
- $D_{D_j}$: Minimal demand of load $j$ (MW).
- $T_{lb}$: Distribution factor of line $l$ with respect to the power injection at bus $b$.

C. Functions and Mappings

- $f_G(\bullet)$: Mapping $f_G : G \rightarrow B$ from the index set of generators $G$ to the index set of buses $B$, indicating the location of generators.
- $f_D(\bullet)$: Mapping $f_D : D \rightarrow B$ from the index set of loads $D$ to the index set of buses $B$, indicating the location of loads.
- $U(\bullet)$: Upper round of a fraction/decimal.
- $C_N^M$: Combination with parameters $M$ and $N$.

D. Lagrange Multipliers

- $\lambda$: Multiplier corresponding to the balance constraint.
- $\mu_l$: Multiplier corresponding to the power flow limit of transmission line $l$.
- $\xi_i$: Multiplier corresponding to the output upper limit of generator $i$.
- $\eta_i$: Multiplier corresponding to the minimal output limit of generator $i$.
- $\xi_{G_i}$: Multiplier corresponding to the maximal demand requirement of the load $j$.
- $\xi_{D_j}$: Multiplier corresponding to the minimal demand requirement of the load $j$.  

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I. INTRODUCTION

The marginal operation costs of many different types of generators are not piecewise linear with generation output [1]. On the contrary, offer/bid curves of wholesale electricity markets are usually stepwise in practice, due to the simplicity of processing such offers compared with other forms, such as piecewise linear or quadratic offers. If the steps/blocks are small enough and the number of blocks is large enough, participants can use stepwise offers to approximate any offer curve or marginal cost curve. However, certain market rules are usually applied to reduce the computation burdens in reality; for example, the 1 MWh/h minimal bid size in Nordpool [2], a maximum of 10 blocks for a bidding offer in PJM [3] and Australia [4], and 10 MWh/h minimal bidding requirements in the balancing markets of Denmark, Sweden, Norway, and Finland [5].

Stepwise bid/offer curves may lead to multiple dual solutions (MDS), i.e., the dual problem of market optimization has multiple solutions. In this situation, spot pricing theory [6] or any standard pricing methodology based on dual solutions fail to yield unique market clearing prices.

Current solutions for MDS include:
- pricing based on 1 MW of additional generation or additional consumption [3], which is a biased allocation of social surplus in favor of the demand side or the generation side;
- mandatory rebidding [4], which involves additional operations and may cause increasing aggressive bidding.

In a large interconnected electricity market with many participants, the increment between price bands is relatively small. Thus, the controversial part of the social surplus is limited and will be shared by a large group of market participants, indicating a limited or negligible effect of MDS. Existing methods such as those mentioned above are not significantly biased, and are thus acceptable in such situations.

In a small electricity market where few participants dominate, for example, the market of a local electric power system of an island [7], the possibility of MDS occurrence is high and the impact is significant.

Based on the characteristics of the electricity market dispatch model, this paper develops a pricing methodology for MDS. The resultant clearing price is unique, consistent with the classical spot pricing theory, and irrespective of uniform pricing or nodal-pricing mechanisms. The solution also possesses important properties such as “boundary requirement”, “consistency requirement”, “neutrality requirement”, “sensitivity requirement”, and “limit appropriateness”, which other solutions do not hold.

II. SPOT PRICING REVISITED AND PROBLEM FORMULATION

A double-sided auction market dispatch problem (including demand-side bidding and transmission constraints) can be formulated as an optimization model as follows [6], [8], [9]:

\[
\begin{align*}
\min_{P_{Gi}, P_{Dj}} & \quad \sum_{i=1}^{N_G} P_{Gi} - \sum_{j=1}^{N_D} P_{Dj} \\
\text{s.t.} & \quad \sum_{i=1}^{N_G} P_{Gi} = \sum_{j=1}^{N_D} P_{Dj}
\end{align*}
\]

(1)

(2)

\[
\sum_{l=1}^{L} \left[ T_{lb} \cdot \sum_{j=1}^{N_G} (P_{Gm} - P_{Dn}) \right] \leq F_l \quad (l = 1, 2, \ldots, L)
\]

(3)

\[
P_{Gj} \leq \bar{P}_{Gj} \leq \bar{P}_{Gj} (i = 1, 2, \ldots, N_G)
\]

(4)

\[
P_{Dj} \leq \bar{P}_{Dj} \leq \bar{P}_{Dj} (j = 1, 2, \ldots, N_D).
\]

(5)

The clearing price of a standard method is determined by Lagrange multipliers of (1)–(5). The Lagrange function can be formulated as follows:

\[
\Gamma = \sum_{i=1}^{N_G} \bar{P}_{Gi} P_{Gi} - \sum_{j=1}^{N_D} \bar{P}_{Dj} P_{Dj} + \lambda \left( \sum_{i=1}^{N_G} N_{Gi} - \sum_{j=1}^{N_D} P_{Dj} \right)
\]

\[
+ \mu_1 \left( \sum_{l=1}^{L} T_{lb} (P_{Gm} - P_{Dm}) - \bar{F}_l \right)
\]

\[
+ \bar{\tau}_i (\bar{P}_{Gj} - \bar{P}_{Gj}) + \bar{\tau}_2 (\bar{P}_{Gj} - \bar{P}_{Gj}) + \cdots + \bar{\tau}_{N_G} (\bar{P}_{Gj} - \bar{P}_{Gj})
\]

\[
- \bar{\tau}_1 (\bar{P}_{Gj} - \bar{P}_{Gj}) - \bar{\tau}_2 (\bar{P}_{Gj} - \bar{P}_{Gj}) - \cdots - \bar{\tau}_{N_G} (\bar{P}_{Gj} - \bar{P}_{Gj})
\]

\[
+ \bar{\xi}_1 (\bar{P}_{Dj} - \bar{P}_{Dj}) + \bar{\xi}_2 (\bar{P}_{Dj} - \bar{P}_{Dj}) + \cdots + \bar{\xi}_{N_D} (\bar{P}_{Dj} - \bar{P}_{Dj})
\]

\[
- \bar{\xi}_1 (\bar{P}_{Dj} - \bar{P}_{Dj}) - \bar{\xi}_2 (\bar{P}_{Dj} - \bar{P}_{Dj}) - \cdots - \bar{\xi}_{N_D} (\bar{P}_{Dj} - \bar{P}_{Dj}).
\]

(6)

The Karush-Kuhn-Tucker (KKT) optimality conditions include

\[
\frac{\partial \Gamma}{\partial \bar{P}_{Gi}} = p_{Gi} + \lambda + \sum_{i=1}^{L} \mu_{T(i)} = 0, \quad \bar{\tau}_i = 0, \quad (i = 1, 2, \ldots, N_G)
\]

(7)

\[
\frac{\partial \Gamma}{\partial \bar{P}_{Dj}} = -p_{Dj} - \lambda - \sum_{i=1}^{L} \mu_{T(i)} = 0, \quad \bar{\xi}_j = 0, \quad (j = 1, 2, \ldots, N_D)
\]

(8)

\[
\sum_{i=1}^{N_G} P_{Gi} - \sum_{j=1}^{N_D} P_{Dj} = 0
\]

(9)

and a set of complementary equations

\[
\mu_1 \left( \sum_{l=1}^{L} T_{lb} (P_{Gm} - P_{Dm}) - \bar{F}_l \right) = 0, \quad (l = 1, 2, \ldots, L)
\]

(10)

\[
\bar{\tau}_i (\bar{P}_{Gi} - \bar{P}_{Gi}) = 0, \quad (i = 1, 2, \ldots, N_G)
\]

(11)

\[
\bar{\tau}_i (\bar{P}_{Gi} - \bar{P}_{Gi}) = 0, \quad (i = 1, 2, \ldots, N_G)
\]

(12)

\[
\bar{\xi}_j (\bar{P}_{Dj} - \bar{P}_{Dj}) = 0, \quad (j = 1, 2, \ldots, N_D)
\]

(13)

\[
\bar{\xi}_j (\bar{P}_{Dj} - \bar{P}_{Dj}) = 0, \quad (j = 1, 2, \ldots, N_D)
\]

(14)

Let the index of the marginal generator (load) be k, according to KKT conditions, we have \(\bar{\tau}_k = \bar{\xi}_k = 0\), therefore, \(p_{Gk} + \lambda + \sum_{i=1}^{L} \mu_{T(i)} = 0\). The standard practice
is to set the nodal market clearing price at bus \( f(k) \) to be
\[
\rho f(k) = -\lambda - \sum_{i=1}^{N_i} \mu_i T_{f0}(i) [6], [8].
\]

Fig. 1 illustrates the uniform pricing cases in which the last
chosen generator/load is cleared between its upper or lower
limits. In these situations, the dual problem of (1)–(5) has a
unique solution. However, in general, both the primal and dual
solutions of market optimization can yield multiple solutions.
In the case where (1)–(5), i.e., the primal problem of market
dispatch, has multiple solutions, implying that more than one
participant appears as clearing price setters, certain proportional
rules can be used to solve the puzzle [11].

Fig. 2 illustrates an example where the dual problem of
(1)–(5) has multiple solutions. In this situation, standard
pricing methods using multipliers cannot yield unique solu-
tions.

III. SUGGESTED METHOD

To address the challenge of price setting under MDS, a new
pricing methodology is proposed. This section is divided into
three parts: in Section III-A, the fundamental properties of
model (1)–(5) are analyzed and critical elements for MDS are
identified. In Section III-B, a solution method using the perspec-
tive of surplus allocation is proposed. Finally, in Section III-C,
the effectiveness and implications of the proposed method are
examined and compared with other methods.

A. Theoretical Interval of MDS Pricing

For the \( j \)th load dispatched at its upper limit (UL Load),
based on (12), we have \( \hat{\gamma}_j = 0 \). Thus, (7) becomes
\[
p_{Gi} + \lambda + \sum_{i=1}^{L} \mu_i T_{f0}(i) + \hat{\gamma}_i = 0. \tag{15}
\]

Because \( \hat{\gamma}_i \geq 0 \)
\[
-\lambda \geq p_{Gi} + \sum_{i=1}^{L} \mu_i T_{f0}(i), \tag{16}
\]

For the \( j \)th load dispatched at its upper limit (UL Load), fol-
lowing the same argument, we have \( \hat{\xi}_j = 0 \) and
\[
-p_{Dj} - \lambda - \sum_{i=1}^{L} \mu_i T_{f0}(i) + \hat{\xi}_j = 0. \tag{17}
\]

Again because \( \hat{\xi}_j \geq 0 \)
\[
-\lambda \leq p_{Dj} + \sum_{i=1}^{L} \mu_i T_{f0}(i), \tag{18}
\]

Similarly, for the \( i \)th generator dispatched at its lower limit
(LL Gen), we have \( p_{Gi} + \lambda + \sum_{i=1}^{L} \mu_i T_{f0}(i) - \hat{\gamma}_i = 0 \) and
\[
-\lambda \leq p_{Gi} + \sum_{i=1}^{L} \mu_i T_{f0}(i). \tag{19}
\]

For the \( j \)th load dispatched at its lower limit (LL Load), we
have \( -p_{Dj} - \lambda - \sum_{i=1}^{L} \mu_i T_{f0}(i) - \hat{\xi}_j = 0 \) and
\[
-\lambda \geq p_{Dj} + \sum_{i=1}^{L} \mu_i T_{f0}(i). \tag{20}
\]

Now define
\[
S_h = \min \left( p_{Gi} + \sum_{i=1}^{L} \mu_i T_{f0}(i), i \in \text{LL Gen} \right),
\]
\[
S_l = \max \left( p_{Gi} + \sum_{i=1}^{L} \mu_i T_{f0}(i), i \in \text{UL Gen} \right),
\]
\[
D_h = \min \left( p_{Dj} + \sum_{i=1}^{L} \mu_i T_{f0}(i), j \in \text{UL Load} \right),
\]
\[
D_l = \max \left( p_{Dj} + \sum_{i=1}^{L} \mu_i T_{f0}(i), j \in \text{LL Load} \right). \tag{21}
\]

\( D_h, S_h, D_l, \) and \( S_l \) are four important values for pricing
under MDS; their counterpart in normal situations is the mar-
ginal offer/bid. For conciseness of presentation, we will call
them pseudo marginal offers/bids (PMBs) in the rest of the paper.
Each PMB consists of a real offer of a specific generator
(load) and a congestion term determined by the transmission
network.

Summarizing formulae (16)–(21), the nodal clearing price at
the reference bus \( \rho = -\lambda \) should fall in the interval defined as
follows:
\[
\max(D_l, S_l) \leq \rho \leq \min(D_l, S_l). \tag{22}
\]

Inequality (22) defines a range of market clearing prices
and the key issue is resolving this multiplicity. For concis-
ness, the left and right sides of (22), i.e., \( \max(D_l, S_l) \) and
\( \min(D_l, S_l) \), are denoted as clearing price lower limit (CPLL)
and clearing price upper limit (CPUL) in the rest of the paper.
and parameters \( \lambda, \mu, \eta, \xi, \xi, \bar{\mu} \) can be determined based on a singleton set \( \mathcal{A} \). Therefore, we should set the clearing price 
\[
\rho = -\lambda = d + \frac{b + c}{a + 2b + c} \cdot (a + b + c).
\] (25)

The proposed pricing is derived from an incentive-compatible surplus allocation perspective. The economic foundation endows the proposed pricing with a group of favorable properties. Section III-C will compare the proposed solution with other possible solutions and show that the proposed pricing satisfies all the five critical properties defining a consistent, impartial, and stable pricing.

Equation (25) provides an additional relationship between multipliers \( \lambda, \mu \) because on the right side, parameters \( a, b, c, \) and \( d \) are determined by the PMBs [as given in (23)] which are determined by multiplier \( \mu \) [as given in (21)], while on the left side, \( \rho = -\lambda \). Thus, with (25) as the additional equation to the KKT conditions, we can now solve the system and obtain unique values of the Lagrange multipliers \( \lambda, \mu, \eta, \xi, \xi, \bar{\mu} \) in the MDS situation. Then, the PMBs \( D_h, S_h, D_t, S_t \) and parameters \( a, b, c, \) and \( d \) can be determined based on a singleton set of Lagrange multipliers \( \lambda, \mu, \eta, \xi, \xi, \bar{\mu} \). Therefore, the nodal market clearing price at the reference bus is uniquely given as in formula (25). While the nodal market clearing price at any given bus [say bus \( f(i) \)] is uniquely determined as
\[
\rho_{f(i)} = -\lambda - \sum_{l=1}^{N_b} \mu_l T_{f(i)}.
\] (26)

A (system-wide) uniform pricing market can be regarded as a specific case of a locational marginal pricing (LMP) market where the transmission constraints are not considered. In the uniform pricing situation, the proposed pricing reduces into a much simpler formulation. Notice that in this situation, the congestion terms in (21) are equal to 0, so the values of pseudo marginal offers/bids \( D_h, S_h, D_t, S_t \) and parameters \( a, b, c, \) and \( d \) can be directly obtained (rather than obtained through solving a set of equations as in an LMP market). Therefore, the uniform clearing price is directly determined by (25).

In most pool-based electricity markets, marginal losses are considered in the LMP calculation. The proposed methodology can be extended to consider the marginal loss pricing.

Let \( P^* = \sum P_{Gi}^* = \sum P_{Di}^* \) be the total cleared demand/supply (obtained by solving the primal problem of market dispatch), we have
\[
PCS = (a + b + c) \times P^*.
\] (24)

Notice that for the accepted bids, if allocate \((a + b) \times P^*\), surplus for them, they have no incentive to strategically lower their bids. Define this surplus as \( CIS \) (incentive surplus for consumer), which is a part of \( PCS \) including \( AS \), we have \( CIS = (a + b) \times P^* \). Symmetrically, if allocate \((b + c) \times P^*\) surplus for the accepted offers, they have no incentive to strategically lift their offers. Define this surplus as \( SIS \) (incentive surplus for supplier), which is a part of \( PCS \) including \( AS \), we have \( SIS = (b + c) \times P^* \). Notice that there is a unique point at which \( PCS \) is allocated in proportion to \( CIS \) and \( SIS \). If the pricing is based on this rationale, the resultant price should be

The objective of market optimization is to maximize the social welfare. Pricing practice can therefore be perceived as distributing the social surplus between consumers and suppliers.

Fig. 3 illustrates one such situation, where \( D_h > S_h \) and \( D_t > S_t \). Note that the dispatch level and the social surplus (SS, total green shaded area) are uniquely determined. Within the SS, the consumer surplus (CS) and supplier surplus (PS) are marked with horizontal stripes and vertical stripes, respectively. There is a controversial part between CS and PS, marked with crossed stripes. Standard pricing based on multipliers allows multiple distributions of this controversial part, as defined by (22). Any possible unique pricing practice can be perceived as distributing this controversial part of the SS between consumers and suppliers. The rest of this paper will denote this part of SS as the allocatable surplus (AS).

Cleared suppliers have incentive to lift their offer prices, until the clearing price arises to a little bit lower than \( \min(D_h, S_h) \) while the cleared consumers have incentive to lower their bid prices, until the clearing price drops to a little bit higher than \( \max(D_t, S_t) \). Therefore, we should set the clearing price for consumers as \( \max(D_t, S_t) \), and set the clearing price for suppliers as \( \min(D_h, S_h) \), so that both sides have no incentive to manipulate their price (and the equilibrium strategy is to offer/bid their marginal cost/utility). However, one unique price is required in the pool-based markets (another problem is, one needs outside resources to carry the above two-price mechanism). Thus, this paper proposes a unique pricing method using the insights from the two-price mechanism.

Readers can notice that the discussion is now extended beyond the AS area. Let us define another important surplus concept, the pseudo common surplus (PCS), illustrated in Fig. 3 as the area within the red line. Note that PCS is different from social surplus. PCS is based on incentive, rather than real surplus. It is a supplementary tool for deciding a unique pricing, thus named “pseudo”. The upper and lower borders of the PCS are PMBs \( D_h \) and \( S_t \). For concise notation, further define
\[
\begin{align*}
\alpha &= D_h - \min(D_h, S_h) \\
b &= \min(D_h, S_h) - \max(D_t, S_t) \geq 0 \\
c &= \max(D_t, S_t) - S_t \\
d &= S_t \geq 0.
\end{align*}
\] (23)

Fig. 3. Social, consumer, supplier, allocatable, and pseudo common surpluses under MDS (the social surplus is represented by the total green shaded area).
As a summary of the above deduction and analysis, the following procedure can be used as a general program to generate unique pricing for a certain power system:

1) Solve the market dispatch problem (1)–(5) by standard LP solver. Get a solution (dispatch). Form KKT conditions and check if the solution can generate a unique set of multipliers. If yes, use standard approach based on the set of multipliers for pricing. End. If no, MDS occurs, go to Step 2) (3) if is not active; go to Step 3) otherwise.

2) Formulate PMBs using (21), formulate parameters \(a, b, c,\) and \(d\) using (23), the uniform clearing price is directly determined by (25). End.

3) Use the dispatch solution to distinguish all the buses into two categories: the MDS buses (on which all the generators and consumers are cleared at their upper or lower limit) and the non-MDS buses (on which at least one generator or consumer are cleared at neither their upper nor their lower limit).

4) Choose one MDS bus as the reference bus and form the KKT conditions of the optimal market dispatch problem (1)–(5). Locate the optimal solution branch of KKT conditions and form a reduced set of equations with Lagrange multipliers.

5) If the number of equations \(N_v\) is equal to the number of variables (multipliers) \(N_v\), go to Step 8); else if \(N_v = N_v = 1\), go to Step 6); else if \(N_v = N_v > 1\) (high-order MDS), go to Step 7).

6) Formulate PMBs using (21), formulate parameters \(a, b, c,\) and \(d\) using (23), formulate the proposed pricing criteria using (25). Thus, we get one additional equation of \(\lambda_k\) and \(\mu_k\) can be formed. Use (26) to generate the nodal price of every bus \(\rho_{f(i)}(\mu_k)\) should be the same with different reference buses; thus, in total, we have \(N_m \cdot C^2_{N_v} = N_m \cdot N_v \cdot (N_v - 1)/2 \geq N_m \cdot (N_v - N_v)\) equations and \(N_m \cdot (N_v - N_v + 1)\) variables. Go to Step 8).

8) Obtain a unique set of KKT multipliers and use the unique set of KKT multipliers to generate unique clearing price at each bus by (26). End.

C. Properties and Comparison

The previous subsection describes the proposed pricing and its economic foundation. The economic foundation endows the proposed pricing with a group of favorable properties other pricing method does not hold. This subsection first discusses these properties, and then compares the proposed pricing with other pricing methods based on these criteria. The five properties discussed in this subsection are:

1) Boundary requirement: the pricing always falls within the interval defined in (22). Otherwise, the pricing does not satisfy the social maximization criteria.

2) Consistency requirement: the pricing should be consistent with the normal LMP under non-MDS conditions. Otherwise, the pricing is incompatible.

3) Neutrality requirement: the pricing should be impartial between supplier and consumer with respect to the surplus allocation. Otherwise, the pricing will cause biased allocation of the social surplus.

4) Sensitivity requirement: the pricing should respond appropriately to changes of offers/bids. Mathematically, the appropriate response implies the boundedness of sensitivities of price to offers/bids; otherwise, a small offer/bid modification will influence the price drastically, making the market highly volatile and vulnerable to participants’ behavior/misbehavior.

5) Limit appropriateness: the trend of the pricing should be appropriate when parameters approach extreme values.

First, let us prove that the proposed pricing satisfies the “boundary requirement”, i.e., prove that \(\rho\) satisfies (22). Based on (22), (23) can be reformulated as

\[c + d \leq \rho \leq b + c + d.\]  (27)

First, we consider the left-side inequality. Based on (25):

\[\rho = c + d + (a + b) \cdot b/a + 2b + c.\]

Then from the right-side inequality:

\[c + b + c + a + 2b + c \geq 0\]

This completes the proof that (27) holds.

The inequality (27) can also be understood in the following way. Notice that in the derivation of above proof, \(\rho\) is formulated as

\[\rho = c + d + \frac{a + b}{a + 2b + c} \cdot b.\]  (28)

Equation (28) can be regarded as dividing AS in proportion to CIS and SIS. The two formulations of \(\rho\) (25) and (28), indicate that the proposed method is not only a CIS-SIS allocation of PCS [as (25) indicates], but also a SIS-CIS allocation of AS [as (28) indicates], where CIS and SIS are the basic elements. The two perspectives (allocating AS and allocating PCS) have achieved agreement under the proposed pricing.

Second, let us examine the “consistency requirement”. Normal (non-MDS) situations can be regarded as a special MDS case in which \(\max(D_t, S_t) = \min(D_t, S_t).\) In this case, \(b = 0\) and (25) simplifies to \(\rho^{\max} = c + d,\) which is exactly the standard LMP formula in the non-MDS situation. In fact, satisfying the “boundary requirement” is a sufficient condition for satisfying the “consistency requirement”.

Third, let us examine the “neutrality requirement”. When \(S_t = D_t\) and \(S_t = D_t,\) the generation side and demand side are symmetrical. In this situation, \(a = c = 0.\) Then, from (25), we have \(\rho = d + b/2,\) which suggests that \(\text{AS}(b \cdot P)\) is equally divided between the generation side and demand side.
Fourth, to examine the sensitivities [12], the partial derivatives of \( \rho \) to parameters \( a, b, c, \) and \( d \) are calculated:

\[
\frac{\partial \rho}{\partial a} = \frac{b(b + c)}{(a + 2b + c)^2} \\
\frac{\partial \rho}{\partial b} = \frac{a(a + 2b + c) + b(a + 2b + 2c)}{(a + 2b + c)^2} \\
\frac{\partial \rho}{\partial c} = 1 - \frac{b(a + b)}{(a + 2b + c)^2} \\
\frac{\partial \rho}{\partial d} = 1.
\]

It can be shown that \( \partial \rho^n / \partial a, \partial \rho^n / \partial b, \partial \rho^n / \partial c, \) and \( \partial \rho / \partial d \) all have upper bounds smaller than or equal to 1 and lower bounds larger than or equal to 0:

\[
\because a, b, c, d \geq 0, \therefore \frac{\partial \rho}{\partial c} \leq 1 \text{ and } \frac{\partial \rho}{\partial a} \geq 0 \\
\frac{\partial \rho}{\partial c} - \frac{\partial \rho}{\partial b} = \frac{(a + b + c)(b + c)}{(a + 2b + c)^2} \geq 0 \\
\frac{\partial \rho}{\partial d} - \frac{\partial \rho}{\partial a} = \frac{(a + b + c)(a + b)}{(a + 2b + c)^2} \geq 0.
\]

Summarizing (33)–(35), we have

\[
1 \geq \frac{\partial \rho}{\partial d} \geq \frac{\partial \rho}{\partial c} \geq \frac{\partial \rho}{\partial b} \geq \frac{\partial \rho}{\partial a} \geq 0.
\]

Equation (36) shows that, for all the possible values \([0, \infty)\) of parameters \( a, b, c, \) and \( d \), the sensitivities of \( \rho \) to parameters \( a, b, c, \) and \( d \) are bounded between \([0, 1]\) and a general uniform ranking of these sensitivities exists.

From (23), we have

\[
\frac{\partial \rho}{\partial D_h} = \begin{cases} 
\frac{\partial \rho}{\partial S_h}, & D_h < S_h \\
\frac{\partial \rho}{\partial \Delta_h}, & D_h \geq S_h
\end{cases}
\]

\[
\frac{\partial \rho}{\partial S_h} = \begin{cases} 
0, & D_h < S_h \\
\frac{\partial \rho}{\partial D_h} - \frac{\partial \rho}{\partial S_h}, & D_h \geq S_h
\end{cases}
\]

\[
\frac{\partial \rho}{\partial D_h} = \begin{cases} 
0, & D_h < S_l \\
\frac{\partial \rho}{\partial D_h} - \frac{\partial \rho}{\partial S_l}, & D_h \geq S_l
\end{cases}
\]

\[
\frac{\partial \rho}{\partial S_l} = \begin{cases} 
\frac{\partial \rho}{\partial D_h} - \frac{\partial \rho}{\partial S_h}, & D_l < S_l \\
\frac{\partial \rho}{\partial D_h} - \frac{\partial \rho}{\partial S_l}, & D_l \geq S_l
\end{cases}
\]

Based on (36) and from (37)–(40), we have: \(0 \leq \partial \rho / \partial D_h \leq 1, \ 0 \leq \partial \rho / \partial S_h \leq 1, \ 0 \leq \partial \rho / \partial D_h \leq 1, \) and \(0 \leq \partial \rho / \partial S_l \leq 1.\) In other words, for all possible values \([0, \infty)\) of the four PMBs, i.e., \(D_h, S_h, D_L, \) and \(S_L,\) the sensitivities of the proposed pricing to PMBs are bounded within \([0, 1].\) Note that the sensitivity of the standard pricing to the marginal bid is one. The marginal influence of any PMB modification on the proposed pricing will be at most equal to the influence of a marginal offer/bid modification on standard pricing. The boundedness of the sensitivities of price to offers/bids ensures that the volatility/vulnerability of the price to participants’ behavior/misbehavior is limited. Section IV will provide case studies of the performance of the proposed pricing to the participants’ offer/bid change.

Lastly, let us examine an extreme scenario in which the last accepted load is price inelastic. In this situation, \(D_h = \infty,\) then \(a = \infty;\) by calculating the limit, we have \(\lim_{a \to \infty} \rho = b + c + d.\) This means that \(AS\) is totally assigned to the generation side. This property characterizes the performance of the proposed solution under extreme/infinite situations. The practical implication can be understood as follows: from the perspective of the demand side, \(b\) can be neglected compared with \(a\) (when \(D_h\) is infinite/extremely large), while from the generation side, \(b\) is still important.

We have now proved that the proposed pricing satisfies the “boundary requirement”, “consistency requirement”, “neutrality requirement”, “sensitivity requirement”, and “limit appropriateness”. In the following, the proposed method will be further examined through comparison with other pricing methods.

Based on the “boundary requirement”, another two price-setting methods can be derived: 1) dividing \(AS\) according to the proportions between \(a\) and \(c,\) which can be formulated as \(\rho^{n-c} = c + d/a + c \cdot b;\) and 2) equally dividing \(AS\) between the demand side and the generation side, which can be formulated as \(\rho^{n-c} = c + d + 1/2 \cdot b.\)

First, let us examine 1). This method is designed according to the “boundary requirement” and “limit appropriateness”, so these two requirements are automatically satisfied. However, \(\rho^{n-c}\) does not satisfy the “neutrality requirement”, because when \(a = c = 0, \rho^{n-c}\) does not exist. In addition, \(\rho^{n-c}\) does not satisfy the “sensitivity requirement” because \(\partial \rho^{n-c} / \partial a\) and \(\partial \rho^{n-c} / \partial c\) are unbounded.

Next, let us examine 2). This method is designed according to the “boundary requirement” and “neutrality requirement”, so these two requirements are automatically satisfied. In addition, the sensitivities of \(\rho^{n-c}\) with regard to parameters \(b, c,\) and \(d\) are all constant, and thus bounded. However, \(\rho^{n-c}\) is irrelevant to parameter \(a,\) so \(\rho^{n-c}\) cannot respond to variations in \(a,\) and therefore does not satisfy “limit appropriateness”.

IV. NUMERICAL EXAMPLES

In this section, two cases will be developed to examine the proposed pricing method. The first case considers a uniform-pricing scenario, while the second case is a nodal-pricing scenario. The case studies will demonstrate that MDS phenomena can occur in both markets. Transmission constraints can actually transform the market optimization from a non-MDS situation to an MDS situation. The proposed method can address the pricing difficulty in all these cases.

A. Case 1: Uniform-Pricing Scenario

In this case, a pool-based market with a double-sided auction is used to examine the proposed method. The generators/loads information for this case is given in Table I.

In this situation, multiple dual solutions exist and classical pricing methods cannot determine the market clearing price.

As shown in Section II-B, the pseudo marginal offers/bids \(D_h, S_h, D_L, \) and \(S_L\) can be obtained directly through (21) in this uniform pricing setting. We thus have: \(S_h = 5, S_l = 2, D_h = 7, D_L = 3.\)

Then by formula (23), we have: \(a = 2, c = 1, b = 2, d = 2.\)

Then by formula (25), we have: \(\rho = a + b + c + d = a + b/a + 2b + c \cdot (a + b + c) \approx 4.143.\)
TABLE I
PARTICIPANTS INFORMATION

<table>
<thead>
<tr>
<th></th>
<th>Capacity/Demand (MW)</th>
<th>Offer/Bid Price ($/MW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>50</td>
<td>1</td>
</tr>
<tr>
<td>G2</td>
<td>100</td>
<td>2</td>
</tr>
<tr>
<td>G3</td>
<td>150</td>
<td>5</td>
</tr>
<tr>
<td>L1</td>
<td>100</td>
<td>8</td>
</tr>
<tr>
<td>L2</td>
<td>50</td>
<td>7</td>
</tr>
<tr>
<td>L3</td>
<td>50</td>
<td>3</td>
</tr>
</tbody>
</table>

Notice that the proposed method has obtained a unique uniform clearing price and the proposed price satisfies the “boundary requirement” [3 ≤ ρ ≤ 5 in this case based on inequality (22)].

In the following part, we will examine the sensitivity of the proposed pricing method subject to offers/bids variations of generators/loads. Figs. 4 and 5 show the response of the proposed price, CPLL, and CPUL to variation of G2’s offer and L2’s bid, respectively.

In Fig. 4, the response curves can be divided into four sections (A-D) as G2’s offer changes. In Section IV-A, G2’s offer is not one of the four pseudo marginal offers/bids, so it has no influence on ρ. In Section IV-B, G2’s offer exceeds G1’s offer and becomes Sf, so ρ monotonically increases as G2’s offer increases. In Section IV-C, G2’s offer is still equal to Sf, and G2’s offer exceeds L1’s bid and becomes CPLL. Therefore, ρ monotonically increases as G2’s offer increases. In Section IV-D, G2’s offer exceeds G3’s offer, so G2’s offer is no longer one of the four pseudo marginal offers/bids, so it has no influence on ρ.

In Fig. 5, the response curves can be divided into five sections as L2’s bid changes. In Sections IV-A, L2’s bid is Df of the four pseudo marginal offers/bids, so ρ increases monotonically as L2’s bid increases. In Sections IV-B, L2’s bid exceeds L3’s bid and becomes CPLL, so ρ monotonically increases as L2’s bid increases. In Section IV-C, L2’s bid exceeds L3’s bid and becomes Df, while L3’s bid becomes Df. In this section, ρ still increases monotonically as L2’s bid increases, but the rate of increase has changed (notice that the ρ curve is convex in Section IV-B but concave in Section IV-C). Meanwhile, L2’s bid exceeds G2’s offer and becomes CPUL, so CPUL has the same behavior as L2’s bid. As L2’s bid rises further, in Section IV-D, it exceeds G3’s offer, and CPUL is fixed as G3’s offer. In Section IV-E, L2’s bid exceeds L1’s bid and is no longer one of the four pseudo marginal offers/bids. Therefore, ρ is not influenced by the variation in L2’s bid.

In this case, a uniform pricing scenario is used to examine the performance of the proposed method. In summary, the following observations can be made.

1) When a participant is a marginal participant or one of the pseudo marginal participants, ρ increases as the participant’s offering/bidding price increases (and the response curve will be sectional if the participant’s role shifts among the four pseudo marginal participants); otherwise, ρ will be irrelevant to this participant. This implies an extension of the classical notion that a marginal offer/bid decides the market price.

2) When the problem changes from an MDS situation (handled by the proposed pricing) to a non-MDS situation (handled by standard pricing) or vice versa, the market clearing price behaves continuously (without a sudden jump or drop).

B. Case 2: Nodal-Pricing Scenario

In this case, an LMP market with double-sided auction is used to examine the proposed method. The network topology is shown in Fig. 6, and generators/loads information is provided in Table II.

The optimal dispatch can be obtained through linear programming (1)–(5). If the transmission limits are large enough, the market clearing price at all buses will be the same, set by the offering price of the last-cleared generator (G31).
Table II
Participants Information

<table>
<thead>
<tr>
<th>Capacity/Demand (MW)</th>
<th>Offer Price ($/MW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>G21</td>
<td>1000</td>
</tr>
<tr>
<td>G31</td>
<td>300</td>
</tr>
<tr>
<td>G32</td>
<td>500</td>
</tr>
<tr>
<td>L11</td>
<td>700</td>
</tr>
<tr>
<td>L12</td>
<td>200</td>
</tr>
<tr>
<td>L13</td>
<td>200</td>
</tr>
</tbody>
</table>

Table III
Cleared output (MW)

<table>
<thead>
<tr>
<th>Without Transmission Constraints</th>
<th>With Transmission Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>G21</td>
<td>1000</td>
</tr>
<tr>
<td>G31</td>
<td>100</td>
</tr>
<tr>
<td>G32</td>
<td>0</td>
</tr>
<tr>
<td>L11</td>
<td>700</td>
</tr>
<tr>
<td>L12</td>
<td>200</td>
</tr>
<tr>
<td>L13</td>
<td>200</td>
</tr>
</tbody>
</table>

When the transmission limits are considered, the resultant dispatch will be different, as shown in Table III, where G21 is restricted from 1000 MW to 600 MW because of the transmission limit of line 12. When transmission constraints are active, model (1)–(5) can still be used to decide the optimal dispatch, but deciding the nodal prices becomes a problem. Because all the dispatched generators/loads are cleared at their maximal level, as we can see in Table III, there is no standard marginal participant.

Following Step 1) of the procedure in Section III-B, the market dispatch is formulated as follows based on model (1)–(5) (Bus3 as the reference bus):

\[
\min_{P_{G21} \leq 600, P_{G31} \leq 600, P_{G32} \leq 300, P_{D11} \leq 700, P_{D12} \leq 200, P_{D13} \leq 200} \quad 15P_{G21} + 20P_{G31} + 35P_{G32} - (60P_{D11} + 40P_{D12} + 25P_{D13}) \quad (41)
\]

subject to:

\[
P_{G21} + P_{G31} + P_{G32} = P_{D11} + P_{D12} + P_{D13} \quad (42)
\]

\[
\frac{1}{3}(P_{D11} + P_{D12} + P_{D13}) + \frac{1}{3}P_{G21} \leq 500 \quad (43)
\]

\[
\frac{2}{3}(P_{D11} + P_{D12} + P_{D13}) - \frac{1}{3}P_{G21} \leq 2000 \quad (44)
\]

\[
\frac{1}{3}(P_{D11} + P_{D12} + P_{D13}) + \frac{2}{3}P_{G21} \leq 2000 \quad (45)
\]

\[
0 \leq P_{G21} \leq 1000 \quad (46)
\]

\[
0 \leq P_{G31} \leq 300 \quad (47)
\]

\[
0 \leq P_{G32} \leq 500 \quad (48)
\]

\[
0 \leq P_{D11} \leq 700 \quad (49)
\]

\[
0 \leq P_{D12} \leq 200 \quad (50)
\]

\[
0 \leq P_{D13} \leq 200. \quad (51)
\]

Since the objective function, equality, and inequality constraints are all linear, model (41)–(51) can be solved using standard linear programming. The solution is \((P_{G21}, P_{G31}, P_{G32}, P_{D11}, P_{D12}, P_{D13})^* = (600, 300, 0, 700, 200, 0)\).

The optimal solution is in the branch in which \(\hat{\gamma}_1 = 0, \hat{\gamma}_2 = 0, \hat{\gamma}_3 = 0, \hat{\xi}_1 = 0, \hat{\xi}_2 = 0, \mu_2 = 0, \) and \(\mu_3 = 0.\) Other nonzero multipliers are bounded with the following reduced system of equations:

\[
15 + \lambda + \frac{1}{3}\mu_1 = 0 \quad (52)
\]

\[
20 + \mu_1 + \xi_1 = 0 \quad (53)
\]

\[
35 + \lambda - \mu_2 = 0 \quad (54)
\]

\[
-60 - \lambda + \frac{1}{3}\mu_1 + \xi_1 = 0 \quad (55)
\]

\[
-40 - \lambda + \frac{1}{3}\mu_1 + \xi_2 = 0 \quad (56)
\]

\[
-25 - \lambda + \frac{1}{3}\mu_1 - \xi_3 = 0. \quad (57)
\]

Notice that in the system (52)–(57), there are seven variables \((\lambda, \mu_1, \xi_1, \xi_2, \xi_3, \mu_2, \mu_3)\) with only six equations; thus, there is no unique solution.

Following Step 3), based on the optimal solution, all the buses are categorized into: the MDS buses—Bus1 and Bus3 and the non-MDS buses—Bus2.

Following Step 4), the Bus3 is chosen as the reference bus for simplicity; thus, the reduced set of equations with Lagrange multipliers are directly obtained through (52)–(57). The other MDS bus Bus1 can also be chosen as the reference bus; later on, we will perform that and show that the results will be the same.

Following Step 5), go to Step 6) because \(N_b - N_c = 1\).

Following Step 6), formulate PMBs based on (21). Because \(\hat{\gamma}_2 \geq 0, \hat{\gamma}_3 \geq 0, \hat{\xi}_1 \geq 0, \hat{\xi}_2 \geq 0, \) and \(\hat{\xi}_3 \geq 0,\) we have \(15 \leq \mu_1 \leq 37.5\) from (52)–(57).

Then

\[
S_h = 35 \quad (58)
\]

\[
S_1 = 15 + \frac{1}{3}\mu_1 \quad (49)
\]

\[
D_h = 40 - \frac{1}{3}\mu_1 \quad (50)
\]

\[
D_l = 25 - \frac{1}{3}\mu_1. \quad (51)
\]

Formulate the intermediate parameters based on (23) with (58), we have

\[
a = 5 - \frac{1}{3}\mu_1 \quad (52)
\]

\[
b = 20 - \frac{1}{3}\mu_1 \quad (53)
\]

\[
c = 0 \quad (54)
\]

\[
d = 15 + \frac{1}{3}\mu_1. \quad (55)
\]

Formulate the additional pricing criterion based on (25) with (59), we have

\[
\lambda = 15 + \frac{1}{3}\mu_1 + \frac{1}{9}(60 - \mu_1) \cdot \frac{(75 - 2\mu_1)}{(45 - \mu_1)} \quad (56)
\]

Thus, we get one additional equation of multipliers. Go to Step 8). Now the seven variables \((\lambda, \mu_1, \xi_1, \xi_2, \xi_3, \mu_2, \mu_3)\)
in (52)–(57) can be solved because it provides the seventh equation. The solution of system (52)–(60) is

\[(\lambda, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\tau}_3, \mu_1)^* = (-27.5, 7.5, 7.5, 20, 0.15, 37.5), \]

(61)

Then by (26), the nodal prices can be obtained. The nodal price results are listed in Table IV, where the results of the uncongested case are also listed for comparison. The results show that the proposed method maintains the locational characteristics of LMP. The clearing prices are different between buses. Because of the congestion of Line12, the price of the load pocket (Bus1) is higher than the prices of the generation nodes (Bus2 and Bus3).

V. CONCLUSIONS

Both primal and dual programs of a market optimization can yield multiple solutions, leading to uncertainty in market dispatching/pricing. Standard pricing methodology based on Lagrange multipliers cannot guarantee unique solutions in MDS cases. Existing resolution practices rely on oversimplified or biased approaches, which are inconsistent with the essence of smart grid revolution. By extending the spot pricing and surplus allocation principle, this paper proposes a new pricing methodology that guarantees a unique clearing price when there are multiple dual solutions. The properties of the proposed method have been analyzed and compared with other pricing practices. The proposed methodology exhibits a group of desirable characteristics and is consistent with the classical spot pricing theory. The case studies further demonstrate the robustness and effectiveness of the proposed method in different situations.

REFERENCES


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