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AN ANALYSIS OF THE POLE-ZERO CANCELLATIONS IN H^∞-OPTIMAL CONTROL PROBLEMS OF THE FIRST KIND*

D. J. N. LIMEBEER† AND Y. S. HUNG‡

Abstract. The aim of this paper is to study the pole-zero cancellations which occur in a class of H^∞-optimal control problems which may be embedded in the configuration of Fig. 1. H^∞ control problems are said to be of the first kind if both $P_{12}(s)$ and $P_{21}(s)$ are square but not necessarily of the same size. It is primarily this class of problems which will concern us here. A general bound on the McMillan degree of all controllers which are stabilizing and lead to a closed loop which satisfies $\|\mathcal{R}(s)\|_\infty \leqq \rho$ (ρ need not be optimal in the L^∞-norm sense) is derived. As illustrated in Fig. 1, $\mathcal{R}(s)$ is the transfer function relating $y_1(s)$ to $u_1(s)$. If the McMillan degree of $P(s)$ in Fig. 1 is n, we show that in the single-loop (SISO) case the corresponding (unique) H^∞-optimal controller never requires more than $n-1$ states. In the multivariable case, there is a continuum of optimal controllers whose McMillan degree satisfies this same bound, although other controllers with higher McMillan degree also exist. The derivation of these bounds require several steps, each of which is of independent system theoretic interest.

Key words. pole-zero cancellations, H^∞-optimal control, approximation theory, Nehari's Theorem, degree bounds

AMS(MOS) subject classification. 93C35

1. Introduction. Figure 1 represents a generalized, or abstract, regulator configuration in which a large class of H^∞-optimal control problems may be embedded. If $P_{12}(s)$ and $P_{21}(s)$ are square, we call the associated problem a problem of the first kind. (Problems of the second kind are characterized by having either $P_{12}(s)$ or $P_{21}(s)$ nonsquare. If both off-diagonal blocks are nonsquare, the problem is said to be of the third kind.) Depending on the specific design situation, the inputs $u_1(s)$ could be references, external disturbances, sensor noise signals or the outputs of models representing unknown plant dynamics. The outputs $y_1(s)$, on the other hand, may be plant outputs, plant inputs or the signals driving plant perturbation models. The H^∞ control problem for Fig. 1 is to minimize the L^∞-norm of $\mathcal{R}(s)$ as $K(s)$ is allowed to range over the set of all stabilizing controllers. The general theory of these problems is now well developed and we refer the reader to the expository articles of Francis and Doyle [10], Doyle et al. [6], Safonov et al. [22] and the numerous references therein for details.

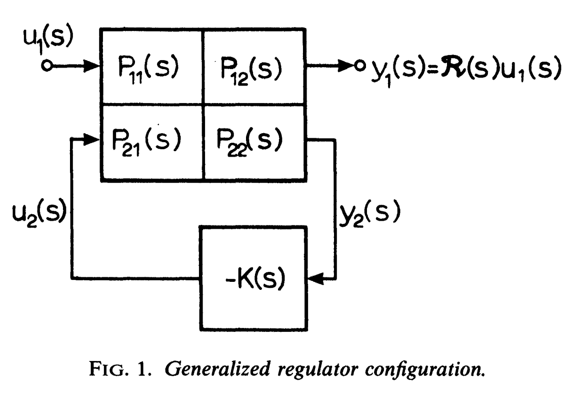

FIG. 1. *Generalized regulator configuration.*

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† Department of Electrical Engineering, Imperial College, London, England.
‡ Electronic and Electrical Engineering Department, University of Surrey, Guildford, Surrey, England.

The purpose of this paper is to carry out a detailed analysis of the cancellation phenomena which occur as a result of H^∞-optimality in the standard regulator configuration mentioned above. Although algorithms for computing these controllers already exist [6], [22], the procedure is so involved that issues such as McMillan degree propagation and the final controller order are obscure. A naive inspection of the procedure may lead one to suspect that the controller degree is several times higher than that of $P(s)$. Since high order controllers are inevitably preceded by computations in high dimensional state-space, expensive and unreliable computations are likely to cause difficulty in complicated design situations. For these reasons, it is our opinion that the complete structural analysis of the computational framework which is presented here will lead to improved computational methods and will also shed light on several aspects of the theory. Our approach is to analyse the entire calculation process in the state-space. This has the advantage of establishing clear links between the theoretical development and existing computer algorithms [6], [22] and also allows one to use Glover's explicit parametrization of all solutions to the Nehari extension problem [12]. This methodology has also been successfully employed in the analysis of cancellation phenomena in H^∞ problems of the second kind. This will be reported on elsewhere [16].

The paper is organized as follows: In § 2 we define notation, describe the problem in specific terms and briefly review the relevant parametrization and optimization theory. Theorem 2.1 is a reformulation of an existing result and it gives an explicit state-space characterization of all the solutions of the L^∞-norm optimization problem in terms of bounded real type equations. In § 3, we establish by way of balancing two Riccati equations associated with the parametrization, the role of the right and left half plane zeros of $P_{12}(s)$ and $P_{21}(s)$. In the case of some problems of the first kind, the lowest achievable infinity norm of the closed loop can be expressed in terms of the solutions to these Riccati equations. Lemma 3.1 and Theorem 3.2 are new results. In § 4, we study the pole-zero cancellations which occur in the closed loop of Fig. 1 when $K(s)$ is chosen to be H^∞-optimal (or suboptimal in a sense to be defined later). This will lead to a general McMillan degree bound for all these controllers. An illustrative example is presented which shows that midcalculation model reduction can produce undesirable effects if done in an ill-advised way. An extension of the McMillan degree bound to the case of minimum entropy controllers is also given in this section. The five results given in § 4 are all believed to be new. Section 5 contains the conclusions. All the proofs have been placed in a series of Appendices.

Some of the proofs involve long calculations. For the reader's convenience, we have written the paper so that no loss of continuity is experienced if these proofs are not studied in the first instance.

2. Notation and background theory.

2.1. Notation.

$\mathbb{R}, \mathbb{C}$	fields of real and complex numbers,
$\mathbb{R}(s)$	field of rational functions in s with real coefficients,
$\mathbb{F}^{m\times l}$	set of $m\times l$ matrices with elements in $\mathbb{F}(=\mathbb{R}, \mathbb{C}, \mathbb{R}(s)$ etc.),
$\mathbb{C}_+, \bar{\mathbb{C}}_+$	open (resp. closed) right half plane,
$\mathbb{C}_-, \bar{\mathbb{C}}_-$	open (resp. closed) left half plane,
$\lambda(A), \lambda_{\max}(A)$	eigenvalues of a square matrix A, largest eigenvalue of A,
A^*	complex conjugate transpose of $A\in\mathbb{C}^{m\times l}$ (transpose if $A\in\mathbb{R}^{m\times l}$),
$\operatorname{In}(A)$	$=(\pi, \nu, \delta)$, the inertia of A, where π, ν and δ are the number of eigenvalues of A in $\mathbb{C}_+, \mathbb{C}_-$ and the $j\omega$ (imaginary) axis,
$A\geqq 0, A>0$	A is positive semidefinite (resp. positive definite),
$A\leqq 0, A<0$	A is negative semidefinite (resp. negative definite),

RL^∞	space of matrices in $\mathbb{R}(s)^{m\times l}$ which have no poles on the $j\omega$ axis (including the point at ∞),
$\|\cdot\|_\infty$	L^∞-norm of matrices in RL^∞,
RH_+^∞, RH_-^∞	subspaces of RL^∞ of matrices which have no poles in $\mathbb{C}_+$ (resp. $\mathbb{C}_-$),
Γ_G	Hankel operator associated with $G(s)\in RH_+^\infty$,
$\sigma_i(G(s))$	ith Hankel singular value of $G(s)$ (i.e. of Γ_G) in decreasing order of magnitude,
$\|G(s)\|_H$	$=\sigma_1(G(s))$, the Hankel norm of $G(s)$,
$\mathrm{Re}\,(s)$, $\bar{s}$, $\|s\|$	the real part, complex conjugate and modulus of $s\in\mathbb{C}$,
$G^*(s)$	$=G(-\bar{s})^*$, the parahermitian conjugate of $G(s)$,
$\Rightarrow, \Leftarrow, \Leftrightarrow$	implies, is implied by, if and only if.

Associated with a transfer function matrix $G(s)\in\mathbb{R}(s)^{m\times l}$ of McMillan degree n is a state-space realization

$$G(s)=D+C(sI-A)^{-1}B \tag{2.1a}$$

where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times l}$, $C\in\mathbb{R}^{m\times n}$, and $D\in\mathbb{R}^{m\times l}$. We will use the alternative notation $G(s)=(A, B, C, D)$ or

$$G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2.1b}$$

In the above notation, we have $G^*(s)=(-A^*, C^*, -B^*, D^*)$ and in the case that D is nonsingular, we also have $G^{-1}(s)=(A-BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})$. If $G^{-1}(s)=G^*(s)$, then $G(s)$ is all-pass. $G(s)$ is called stable (asymptotically stable) if it has no poles in $\mathbb{C}_+$ (resp. $\bar{\mathbb{C}}_+$).

If $G(s)=(A, B, C, D)$ the system matrix corresponding to the given realization is defined as [19]

$$\left[\begin{array}{c|c} sI-A & -B \\ \hline C & D \end{array}\right]$$

and the system zeros are defined to be the points at which the system matrix loses normal rank. In the case when D is nonsingular, the system zeros are also given by $\lambda(A-BD^{-1}C)$. The input decoupling zeros (uncontrollable modes) are points at which $[sI-A \quad B]$ loses rank. The output decoupling zeros (unobservable modes) are the points at which $[sI-A^* \quad C^*]$ loses rank. In the sequel, the term "zero" refers to "system zero" unless stated otherwise. Obviously, {input decoupling zeros} $\cup$ {output decoupling zeros} is a subset of both $\lambda(A)$ and the set of system zeros. The realization (A, B, C, D) is minimal if it has no input/output decoupling zeros. A sufficient condition for this is that all system zeros are distinct from $\lambda(A)$.

If $G_1(s)=(A_1, B_1, C_1, D_1)$ and $G_2(s)=(A_2, B_2, C_2, D_2)$ then the cascade system $G_1G_2(s)$ has a realization given by

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_2 & D_2 \end{array}\right]\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]=\left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right], \tag{2.2}$$

where we have taken the "multiplication" of two realizations to mean cascading the two systems. This is not to be confused with ordinary matrix multiplication. The context will always make the distinction between these two possible interpretations clear.

If a basis change T is introduced into the state-space of $G(s)$, we will take this to mean $G(s)=(TAT^{-1}, TB, CT^{-1}, D)$. The McMillan degree of $G(s)$ will be written as deg (G) and the set of poles of $G(s)$ will be denoted {poles of G}.

Let $P(s)$ be a partitioned matrix with a state-space realization given by

$$P(s)=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \tag{2.3}$$

then

$$P_{ij}(s)=C_i(sI-A)^{-1}B_j+D_{ij} \tag{2.4}$$

is a state-space realization of $P_{ij}(s)$. A linear fractional transformation of the partitioned matrix P and a matrix K is defined as

$$F_l(P,K)=P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}$$

where K is of dimension $l\times m$ if P_{22} has dimension $m\times l$.

2.2. Problem description. Consider the generalized regulator configuration illustrated in Fig. 1. From the equations governing this diagram we see that the transfer function relating y_1 to u_1 is given by

$$\begin{aligned}\mathcal{R}(s)&=F_l(P,-K)\\ &=P_{11}-P_{12}K(I+P_{22}K)^{-1}P_{21}.\end{aligned}$$

We seek to bound the McMillan degrees of all the compensators $K(s)$ which simultaneously achieve an internally stable closed loop and minimize $\|\mathcal{R}(s)\|_\infty$. Throughout this paper we will assume that $P_{12}(s)$ and $P_{21}(s)$ are square (although not necessarily of the same size). We also assume that both D_{12} and D_{21} are nonsingular and that $P_{12}(s)$ and $P_{21}(s)$ have no zeros on the imaginary axis.

It is worth noting that two particular H^∞-optimal control problems which have already received particular attention can be posed in the above setting. The first is the optimal sensitivity problem which has been analysed by Zames and others [3], [7], [8], [21], [27], [28]. In this case we wish to minimize the L^∞-norm of a weighted sensitivity operator given by

$$\mathcal{R}_s(s)=[W_2(I+GK)^{-1}W_1](s) \tag{2.5}$$

where $G(s)$ is the transfer function of a square plant, and $W_1(s)$ and $W_2(s)$ are weighting matrices. If we put

$$P_s(s)=\left[\begin{array}{c|c} W_2W_1 & W_2G \\ \hline W_1 & G \end{array}\right](s), \tag{2.6}$$

then direct calculation shows that

$$\mathcal{R}_s(s)=F_l(P_s(s),-K(s)).$$

The second problem is the optimal robustness problem which has been studied by Glover [13] and Kimura [15]. In the case of optimal robustness with respect to additive perturbations to the plant transfer function, we wish to minimize the L^∞-norm of

$$\mathcal{R}_a(s)=[W_2K(I+GK)^{-1}W_1](s). \tag{2.7}$$

It can be readily shown that if we set

$$P_a(s)=\left[\begin{array}{c|c} 0 & W_2 \\ \hline -W_1 & G \end{array}\right](s) \tag{2.8}$$

then

$$\mathcal{R}_a(s) = F_l(P_a(s), -K(s)).$$

In the sequel, we will study the general class of problems of the first kind and establish common pole-zero cancellation properties which are shared by the specific problems just mentioned.

2.3. Review of H^∞-optimization theory. The solution of H^∞-optimal control problems may be subdivided into two distinct steps. In the first, all the compensators which lead to an internally stable closed loop in Fig. 1 are parametrized. The second step then identifies a subclass of stabilizing compensators which minimize $\|\mathcal{R}(s)\|_\infty$ or else satisfy $\|\mathcal{R}(s)\|_\infty \leqq \rho \in \mathbb{R}$. In the following sections, we will briefly describe the calculations involved in these two steps.

2.3.1. Parametrization of all stabilizing controllers. Let $P(s)$ in Fig. 1 be given by (2.3) and suppose that (A, B_2, C_2) is stabilizable and detectable. Under these conditions $K(s)$ stabilizes the feedback system in Fig. 1 if and only if it stabilizes $P_{22}(s)$. Further, such stabilizing compensators always exist [6], [22]. Let

$$P_{22}(s) = N_r(s)D_r^{-1}(s) = D_l^{-1}(s)N_l(s) \tag{2.9}$$

be right and left rational coprime fractional factorizations of $P_{22}(s)$ and

$$\begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix}(s)\begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix}(s) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.10}$$

the corresponding Bezout identities. All the matrices in (2.10) belong to RH_+^∞ and the set of all compensators which stabilize $P_{22}(s)$, and thus also $P(s)$, are given by [4], [26]

$$K(s) = (U_l + D_rQ)(V_l - N_rQ)^{-1}(s) \tag{2.11}$$

$$= (V_r - QN_l)^{-1}(U_r + QD_l)(s) \tag{2.12}$$

in which the indicated inverses are assumed to exist and $Q(s) \in RH_+^\infty$. It is easy to verify that

$$K(I + P_{22}K)^{-1}(s) = (U_l + D_rQ)D_l(s). \tag{2.13}$$

Hence

$$\begin{aligned} \mathcal{R}(s) &= [P_{11} - P_{12}K(I + P_{22}K)^{-1}P_{21}](s) \\ &= [(P_{11} - P_{12}U_lD_lP_{21}) - (P_{12}D_r)Q(D_lP_{21})](s) \\ &= [T_{11} - T_{12}QT_{21}](s) \end{aligned} \tag{2.14}$$

where the $T_{ij}(s)$ are defined in an obvious way. Equation (2.14) shows that $\mathcal{R}(s)$ is parametrized linearly in $Q(s)$. Since $\mathcal{R}(s) \in RH_+^\infty$ if and only if $Q(s) \in RH_+^\infty$, we would expect that T_{11}, T_{12} and T_{21} all belong to RH_+^∞.

Since (A, B_2) is stabilizable, there exists a state feedback matrix F such that $A - B_2F$ is stable. Similarly, since (A, C_2) is detectable there exists an output injection matrix H such that $A - HC_2$ is stable. Given any such pair of stabilizing matrices F and H, the right and left coprime factorizations of P_{22} together with the solutions of the Bezout identities are given by [6], [18]

$$\begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix}(s) = \left[\begin{array}{c|cc} A - B_2F & B_2 & H \\ \hline -F & I & 0 \\ C_2 - D_{22}F & D_{22} & I \end{array}\right] \tag{2.15}$$

and

$$\begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix}(s) = \left[\begin{array}{c|cc} A-HC_2 & B_2-HD_{22} & H \\ \hline F & I & 0 \\ -C_2 & -D_{22} & I \end{array}\right]. \tag{2.16}$$

Using (2.11) and (2.12), it can be verified by direct calculation that the family of all stabilizing compensators can be parametrized in terms of the linear fractional transformation [6]

$$K(s) = F_l(K_0(s), Q(s)) \tag{2.17}$$

where

$$\begin{aligned} K_0(s) &= \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}(s) = \begin{bmatrix} V_r^{-1}U_r & V_r^{-1} \\ V_l^{-1} & V_l^{-1}N_r \end{bmatrix}(s) \\ &= \left[\begin{array}{c|cc} A-B_2F-HC_2+HD_{22}F & -H & B_2-HD_{22} \\ \hline -F & 0 & I \\ C_2-D_{22}F & I & D_{22} \end{array}\right]. \end{aligned} \tag{2.18}$$

A routine computation will show that the realization of $K_0(s)$ in (2.18) is minimal if (A, B_2, C_2) is minimal.

In order to simplify later calculations, it is helpful at this point to scale (2.3) by replacing it with

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2S_1 \\ \hline C_1 & D_{11} & D_{12}S_1 \\ S_2C_2 & S_2D_{21} & S_2D_{22}S_1 \end{array}\right], \tag{2.19}$$

in which $S_1 = D_{12}^{-1}$ and $S_2 = D_{21}^{-1}$. From now on we will assume that $P(s)$ has been scaled so that both the $(1, 2)$ and $(2, 1)$ blocks of the D-matrix are identities; we therefore assume that the S_i's have already been absorbed into B_2, C_2 and D_{22}. Such an assumption does not incur any loss of generality in our development because the effect of any prescaling on the compensator to bring $P(s)$ into the form of (2.19) may be reversed by replacing $K(s)$ with $S_1K(s)S_2$ at the end of the design process [22]. We now make the following specific choices of the pair of stabilizing matrices F and H, as was suggested by Doyle et al. [6]:

$$F = C_1 + B_2^*X \tag{2.20}$$

where X is the unique positive semidefinite stabilizing solution to the algebraic Riccati equation

$$X(A-B_2C_1)+(A-B_2C_1)^*X - XB_2B_2^*X = 0 \tag{2.21}$$

and

$$H = B_1 + YC_2^* \tag{2.22}$$

where Y is the unique positive semidefinite stabilizing solution to the algebraic Riccati equation

$$Y(A-B_1C_2)^*+(A-B_1C_2)Y - YC_2^*C_2Y = 0. \tag{2.23}$$

It can be shown that the $T_{ij}(s)$ of (2.14) are then given by

$$T(s)=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix}(s)=\left[\begin{array}{cc|cc} A-B_2F & B_2F & B_1 & B_2 \\ 0 & A-HC_2 & -YC_2^* & 0 \\ \hline -B_2^*X & F & D_{11} & I \\ 0 & C_2 & I & 0 \end{array}\right]. \tag{2.24}$$

Note that $T_{12}(s)$, $T_{21}(s)$ and $T_{11}(s)$ belong to RH_+^∞ as expected. Further, F and H have been chosen to make $T_{12}(s)$ and $T_{21}(s)$ inner [6].

2.3.2. Parametrization of all H^∞-optimal controllers. In this section we briefly review the parametrization of all optimal $Q(s)\in RH_+^\infty$ which solve the minimization problem

$$\min \|(T_{11}-T_{12}QT_{21})(s)\|_\infty = \|T_{12}^*T_{11}T_{21}^*(-s)\|_H, \qquad Q\in RH_+^\infty \tag{2.25}$$

or suboptimal $Q(s)\in RH_+^\infty$ which satisfy

$$\|(T_{11}-T_{12}QT_{21})(s)\|_\infty \leqq \rho \in \mathbb{R} \tag{2.26}$$

for some given $\rho > \|T_{12}^*T_{11}T_{21}^*(s)\|_H$.

Due to the norm-preserving properties of the inner matrices $T_{12}(s)$ and $T_{21}(s)$, we may write [21]

$$\|(T_{11}-T_{12}QT_{21})(s)\|_\infty = \|(T_{12}^*T_{11}T_{21}^*-Q)(s)\|_\infty. \tag{2.27}$$

From (2.24), we obtain

$$T_{12}^*T_{11}T_{21}^*(s)=\left[\begin{array}{cc|c} -(A-B_2F)^* & X(B_2D_{11}-B_1)C_2Y & X(B_1-B_2D_{11}) \\ 0 & -(A-HC_2)^* & -C_2^* \\ \hline -B_2^* & (F-D_{11}C_2)Y & D_{11} \end{array}\right] \tag{2.28}$$

which shows that $T_{12}^*T_{11}T_{21}^*(s)\in RH_-^\infty$. Setting $T_{12}^*T_{11}T_{21}^*(s)=X^*(s)$ we get

$$\|(T_{11}-T_{12}QT_{21})(s)\|_\infty = \|X(s)-Q^*(s)\|_\infty \tag{2.29}$$

which turns (2.25) into a multivariable version of the Nehari extension problem [17]. We will call any $Q(s)RH_+^\infty$ which satisfies (2.26) a ρ-suboptimal extension. Glover has shown that all $Q^*(s)$ which satisfy (2.25) or (2.26) may be generated by means of a balanced realization of $X(s)$. In [12], the characterization of all $Q^*(s)$ in the general nonsquare case is given in terms of a linear fractional transformation of transfer function matrices (see [12, Thm. 8.7]). We will however need a state-space version of this characterization in order to derive the main results in § 4. This is stated in the next theorem and a proof which makes use of [12, Thm. 8.7] is given in Appendix A.

THEOREM 2.1. *Let $X(s)=(A, B, C, D)$ be a stable, minimal and balanced realization with Hankel singular values*

$$\sigma_1 \geqq \sigma_2 \geqq \cdots \geqq \sigma_k > \sigma_{k+1} = \sigma_{k+2} = \cdots = \sigma_{k+r} > \sigma_{k+r+1} \geqq \cdots \geqq \sigma_n > 0.$$

Assume that the Hankel singular values have been arranged so that the gramians are given by

$$\operatorname{diag}(\Sigma, \sigma_{k+1}I_r) \tag{2.30}$$

where

$$\Sigma = \operatorname{diag}(\sigma_1, \cdots, \sigma_k, \sigma_{k+r+1}, \sigma_n)$$

and let (A, B, C) *be partitioned conformally with* (2.30)

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C=[C_1 \quad C_2].$$

Also let

$$\Gamma=\Sigma^2-\sigma_{k+1}^2 I. \tag{2.31}$$

Then for any error system $E(s)=X(s)-\hat{X}(s)-Q^*(s)$ *with*

(a) $\|E(j\omega)\|_\infty \leqq \sigma_{k+1}$;

(b) $\hat{X}(s)$ *is stable of McMillan degree k and* $Q^*(s)$ *is totally unstable there exists* $\hat{A}$, $\hat{B}$, $\hat{C}$ *and* $\hat{D}$ *such that* $E(s)$ *has a realization*

$$E(s)=\left[\begin{array}{cccc|c} A_{11} & A_{12} & 0 & 0 & B_1 \\ A_{21} & A_{22} & 0 & 0 & B_2 \\ 0 & 0 & \begin{array}{c}\Gamma^{-1}(\sigma_{k+1}^2 A_{11}^* + \Sigma A_{11}\Sigma \\ -\sigma_{k+1} C_1^*\hat{D}B_1^*)\end{array} & \Gamma^{-1}C_1^*\hat{C} & \Gamma^{-1}(\Sigma B_1+\sigma_{k+1}C_1^*\hat{D}) \\ 0 & 0 & -\sigma_{k+1}\hat{B}B_1^* & \hat{A} & \sigma_{k+1}\hat{B} \\ \hline C_1 & C_2 & -(C_1\Sigma+\sigma_{k+1}\hat{D}B_1^*) & \hat{C} & \sigma_{k+1}\hat{D} \end{array}\right] =: \left[\begin{array}{cc} A_e & B_e \\ C_e & D_e \end{array}\right] \tag{2.32}$$

which satisfies both

$$\begin{bmatrix} -(A_eP_e+P_eA_e^*+B_eB_e^*) & -(B_eD_e^*+P_eC_e^*) \\ -(D_eB_e^*+C_eP_e) & \sigma_{k+1}^2 I-D_eD_e^* \end{bmatrix} = \begin{bmatrix} L_e \\ W_e \end{bmatrix} [L_e^* | W_e^*], \tag{2.33}$$

and

$$\begin{bmatrix} -(A_e^*Q_e+Q_eA_e+C_e^*C_e) & -(C_e^*D_e+Q_eB_e) \\ -(D_e^*C_e+B_e^*Q_e) & \sigma_{k+1}^2 I-D_e^*D_e \end{bmatrix} = \begin{bmatrix} L_{ed}^* \\ W_{ed}^* \end{bmatrix} [L_{ed} | W_{ed}] \tag{2.34}$$

where in the above equations

$$P_e=\begin{bmatrix} \Sigma & 0 & I & 0 \\ 0 & \sigma_{k+1}I_r & 0 & 0 \\ I & 0 & \Sigma\Gamma^{-1} & 0 \\ 0 & 0 & 0 & \sigma_{k+1}^2\hat{P} \end{bmatrix}, \quad Q_e=\begin{bmatrix} \Sigma & 0 & -\Gamma & 0 \\ 0 & \sigma_{k+1}I_r & 0 & 0 \\ -\Gamma & 0 & \Sigma\Gamma & 0 \\ 0 & 0 & 0 & \hat{Q} \end{bmatrix}$$

and

$$L_e^*=[0\; 0\;\; \sigma_{k+1}\hat{W}^*C_1\Gamma^{-1}\;\; \sigma_{k+1}\hat{L}^*], \qquad L_{ed}=[0\; 0\;\; -\sigma_{k+1}\hat{W}_dB_e^*\;\; \hat{L}_d] \tag{2.35a}$$

for some $\hat{L}$, $\hat{L}_d$, $\hat{W}$, $\hat{W}_d$, $\hat{P}=\hat{P}^*<0$, $\hat{Q}=\hat{Q}^*<0$ *and further*

$$W_e=\sigma_{k+1}\hat{W}, \quad W_{ed}=\sigma_{k+1}\hat{W}_d \quad \text{and} \quad P_eQ_e=\sigma_{k+1}^2 I. \tag{2.35b}$$ □

Remark 2.1. Note that (2.33) and (2.34) are reminiscent of the state-space characterization of bounded real matrices [1], as one would expect because of the condition (a) on $E(s)$. We note however that $E(s)$ is in general not bounded real in the strict sense since it may contain unstable poles.

Remark 2.2. When giving a bounded real type state-space characterization of ρ-suboptimal extensions we make use of the idea given in [12, Remark 8.4]. In this case (2.33) and (2.34) remain in force, $\Gamma=\Sigma^2-\rho^2 I$ replaces (2.31) (with $\sigma_k>\rho>\sigma_{k+1}$),

$$E(s)=\left[\begin{array}{ccc|c} A & 0 & 0 & B \\ 0 & \Gamma^{-1}(\rho^2A^*+\Sigma A\Sigma-\rho C^*\hat{D}B^*) & \Gamma^{-1}C^*\hat{C} & \Gamma^{-1}(\Sigma B+\rho C^*\hat{D}) \\ 0 & -\rho\hat{B}B^* & \hat{A} & \rho\hat{B} \\ \hline C & -(C\Sigma+\rho\hat{D}B^*) & \hat{C} & \rho\hat{D} \end{array}\right] \tag{2.36}$$

replaces (2.32),

$$P_e=\begin{bmatrix}\Sigma & I & 0\\ I & \Sigma\Gamma^{-1} & 0\\ 0 & 0 & \rho^2\hat{P}\end{bmatrix},\qquad Q_e=\begin{bmatrix}\Sigma & -\Gamma & 0\\ -\Gamma & \Sigma\Gamma & 0\\ 0 & 0 & \hat{Q}\end{bmatrix}\tag{2.37}$$

and

$$L_e^*=[0 \quad \rho\hat{W}^*C\Gamma^{-1} \quad \rho\hat{L}^*],\qquad L_{ed}=[0 \quad -\rho\hat{W}_dB^* \quad \hat{L}_d],$$

replace (2.35a) and $W_e=\rho\hat{W}$, $W_{ed}=\rho\hat{W}_d$ and $P_eQ_e=\rho^2I$ replace (2.35b).

For readers who are interested in following through the proof for the ρ-suboptimal case, (A.4) in Appendix A should be replaced by

$$\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}(s)=\left[\begin{array}{c|cc}\Gamma^{-1}(\rho^2A^*+\Sigma A\Sigma) & \Gamma^{-1}\Sigma B & -\Gamma^{-1}C^*\\ \hline C\Sigma & D & I\\ -\rho B^* & \rho I & 0\end{array}\right].\tag{2.38}$$

Conditions (A.5) and (A.6) remain valid but (A.7) no longer applies.

Remark 2.3. Theorem 2.1 and Remark 2.2 are more general than we need in H^∞-optimal control problems since they give a state-space characterization of all error systems associated with Hankel norm approximation problems whereas we are only interested in optimal anticausal (or Nehari) type approximations. Specifically, we will make use of Theorem 2.1 (or Remark 2.2) with $k=0$ (zeroth order Hankel approximation) so that $\hat{X}(s)=0$ and $Q(s)$ becomes a Nehari extension (or ρ-suboptimal extension) of $X(s)$. Also $\|E(s)\|_\infty=\sigma_1$ or $\|E(s)\|_\infty\leqq\rho$ (where $\rho>\sigma_1$) for the optimal or ρ-suboptimal case respectively. □

COROLLARY 2.2. *In the notation of Theorem* 2.1 *and Remark* 2.2 *let* $k=0$ *and* $U(s)=[\hat{A},\hat{B},\hat{C},\hat{D}]$ (*see* (2.32) *and the proof of Theorem* 2.1). *Then*

(i) If $Q(s)$ *is a Nehari extension of* $X(s)$,

$$\deg(Q)\leqq\deg(X)-r+\deg(U)\tag{2.39}$$

where r is the multiplicity of the largest Hankel singular value of X;

(ii) *If* $Q(s)$ *is a* ρ*-suboptimal extension of* $X(s)$,

$$\deg(Q)\leqq\deg(X)+\deg(U).\tag{2.40}$$ □

This corollary follows immediately from an inspection of (2.32) and (2.36).

Remark 2.4. For $k=0$, Theorem 2.1 characterizes all

$$E(s)=X(s)-Q^*(s)\tag{2.41}$$

satisfying conditions (a) and (b). At certain points in the sequel, it is more convenient to work with a realization for $E^*(s)=X^*(s)-Q(s)$ instead of (2.41). For this purpose, we remark here that the form of the bounded real type equations (2.33) and (2.34) is invariant under parahermitian conjugation. It is easy to see that if $E(s)\to E^*(s)$, then we only need to perform the following substitutions in (2.33) and (2.34)

$$A_e\to -A_e^*,\quad B_e\to C_e^*,\quad C_e\to -B_e^*,$$
$$D_e\to D_e^*,\quad P_e\to -Q_e,\quad Q_e\to -P_e.$$

3. Balancing Riccati equations. In this section we will establish some preliminary results which will be needed in the later analysis.

It is well known that the H^∞-optimization problem given by (2.25), or equivalently (2.29), is equivalent to a matrix version of the classical Nevanlinna–Pick interpolation problem, the set of interpolation points being the right half plane zeros of $T_{12}(s)$ and $T_{21}(s)$. Since $T_{12}(s)$ and $P_{12}(s)$, and $T_{21}(s)$ and $P_{21}(s)$ have the same zeros, it is clear that the set of interpolation points is

$$\{\text{zeros of } P_{12}(s) \text{ in } \mathbb{C}_+\} \cup \{\text{zeros of } P_{21}(s) \text{ in } \mathbb{C}_+\}.$$

In this section we will bring this issue into sharp focus by balancing the two Riccati equations (2.21) and (2.23). Furthermore, we will show that the number of right half plane zeros of $P_{12}(s)$ and $P_{21}(s)$ are given, respectively, by the ranks of the solutions X and Y to these two equations.

Consider a change of basis T in the state-space of $P(s)$ in (2.19). In this new basis, $P(s)$ becomes

$$P(s) = \left[\begin{array}{c|cc} TAT^{-1} & TB_1 & TB_2 \\ \hline C_1T^{-1} & D_{11} & I \\ C_2T^{-1} & I & D_{22} \end{array}\right]$$

and the algebraic Riccati equation (2.21) becomes

$$X(TAT^{-1} - TB_2C_1T^{-1}) + (TAT^{-1} - TB_2C_1T^{-1})^*X - XTB_2B_2^*T^*X = 0 \tag{3.1}$$

or equivalently

$$T^*XT(A - B_2C_1) + (A - B_2C_1)^*T^*XT - T^*XTB_2B_2^*T^*XT = 0. \tag{3.2}$$

This shows that the effect of the basis change on X is the congruence transformation

$$X \to T^{-*}XT^{-1} \tag{3.3}$$

where T^{-*} denotes $(T^*)^{-1}$. Similarly, in the new basis, equation (2.23) becomes

$$T^{-1}YT^{-*}(A - B_1C_2)^* + (A - B_1C_2)T^{-1}YT^{-*} - T^{-1}YT^{-*}C_2^*C_2T^{-1}YT^{-*} = 0 \tag{3.4}$$

and hence the effect of the basis change is

$$Y \to TYT^*. \tag{3.5}$$

(Incidentally, in problems of the second and third kind, (3.1) and/or (3.4) have an additional constant term. This term makes no difference to the balancing arguments.)

Combining (3.3) and (3.5) we obtain

$$YX \to TYXT^{-1} \tag{3.6}$$

and it is immediate from (2.20) and (2.22) that

$$F \to FT^{-1}, \tag{3.7}$$

$$H \to TH. \tag{3.8}$$

Condition (3.6) shows that $\lambda(YX)$ are invariant under basis changes in the state-space of $P(s)$. Conditions (3.3) and (3.5), together with $X = X^* \geqq 0$ and $Y = Y^* \geqq 0$, suggest that we may use the construction in [12, Appendix B] to find a basis change T so that in the new basis

$$Y = \begin{bmatrix} \tilde{\Sigma}_1 & & & \\ & \tilde{\Sigma}_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.9}$$

and

$$X=\begin{bmatrix}\tilde{\Sigma}_1 & & & \\ & 0 & & \\ & & \tilde{\Sigma}_3 & \\ & & & 0\end{bmatrix} \tag{3.10}$$

are balanced diagonal matrices. The balancing of the positive definite solutions of standard LQG type Riccati equations has been studied by Joncheere and Silverman [14]. For convenience of analysis we introduce a permutation matrix J such that

$$JXJ^*=\begin{bmatrix}\tilde{\Sigma}_1 & & & \\ & \tilde{\Sigma}_3 & & \\ & & 0 & \\ & & & 0\end{bmatrix}=\begin{bmatrix}\Sigma_2 & 0\\ 0 & 0\end{bmatrix}. \tag{3.11}$$

Clearly $\Sigma_1>0$ and $\Sigma_2>0$.

For notational simplicity, we will absorb the coordinate transformation matrix T into the state-space matrices and rewrite $TAT^{-1}, TB_2, \cdots$ of (3.1) and (3.4) as $A, B_2, \cdots$. If we set $M=A-B_1C_2$ and $C_2=(C_{21}|C_{22})$, where the partitioning is consistent with that in (3.9), we obtain from (2.23)

$$\begin{aligned}&\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}M_{11}^* & M_{21}^*\\ M_{12}^* & M_{22}^*\end{bmatrix}+\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}\\ &\qquad -\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}C_{21}^*\\ C_{22}^*\end{bmatrix}[C_{21}\quad C_{22}]\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}=0.\end{aligned} \tag{3.12}$$

The $(1,1)$ block of (3.12) yields

$$\Sigma_1M_{11}^*+M_{11}\Sigma_1-\Sigma_1C_{21}^*C_{21}\Sigma_1=0, \tag{3.13}$$

that is,

$$M_{11}-\Sigma_1C_{21}^*C_{21}=-\Sigma_1M_{11}^*\Sigma_1^{-1}. \tag{3.14}$$

From the $(1,2)$ block of (3.12) one obtains

$$\Sigma_1M_{21}^*=0\Rightarrow M_{21}=0. \tag{3.15}$$

Making use of (2.22) and what has been deduced above,

$$\begin{aligned}A-HC_2&=M-YC_2^*C_2\\ &=\begin{bmatrix}M_{11}-\Sigma_1C_{21}^*C_{21} & M_{12}-\Sigma_1C_{21}^*C_{22}\\ 0 & M_{22}\end{bmatrix}\\ &=\begin{bmatrix}-\Sigma_1M_{11}^*\Sigma_1^{-1} & M_{12}-\Sigma_1C_{21}^*C_{22}\\ 0 & M_{22}\end{bmatrix}.\end{aligned} \tag{3.16}$$

Applying [12, Thm. 3.3(2)] to (3.13) establishes the implication

$$\delta(\Sigma_1^{-1})=0\Rightarrow 0=\pi(-\Sigma_1^{-1})\geqq \nu(M_{11}). \tag{3.17}$$

Since we have assumed that $P_{21}(s)$ has no zeros on the imaginary axis, $\delta(M_{11})=0$ and therefore

$$\mathrm{In}\,(M_{11})=(\mathrm{rank}\,(Y),0,0). \tag{3.18}$$

Note that $\{\lambda(M_{11})\} \equiv \{\text{right half plane zeros of } P_{21}(s)\}$. Since $A-HC_2$ is asymptotically stable so too is M_{22}.

Defining $Z=J(A-B_2C_1)J^*$ and $(JB_2)^*=\hat{B}_2^*=[\hat{B}_{12}^*|\hat{B}_{22}^*]$ we obtain from (2.21)

$$\begin{bmatrix}\Sigma_2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{bmatrix}+\begin{bmatrix}Z_{11}^* & Z_{21}^*\\ Z_{12}^* & Z_{22}^*\end{bmatrix}\begin{bmatrix}\Sigma_2 & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}\Sigma_2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{B}_{12}\\ \hat{B}_{22}\end{bmatrix}[\hat{B}_{12}^*|B_{22}^*]\begin{bmatrix}\Sigma_2 & 0\\ 0 & 0\end{bmatrix}=0. \tag{3.19}$$

By an argument similar to the one given for (3.12), we have that

$$Z_{11}-\hat{B}_{12}\hat{B}_{12}^*\Sigma_2=-\Sigma_2^{-1}Z_{11}^*\Sigma_2, \tag{3.20}$$

$$Z_{12}=0, \tag{3.21}$$

$$\mathrm{In}\,(Z_{11})=(\mathrm{rank}\,(X),0,0), \tag{3.22}$$

$$J(A-B_2F)J^*=\begin{bmatrix}-\Sigma_2^{-1}Z_{11}^*\Sigma_2 & 0\\ Z_{21}-\hat{B}_{22}\hat{B}_{12}^*\Sigma_2 & Z_{22}\end{bmatrix}, \tag{3.23}$$

and that Z_{22} is asymptotically stable. The eigenvalues of Z_{11} are the right half plane zeros of $P_{12}(s)$. Next, we partition the matrices

$$J[B_1|B_2]=[\hat{B}_1|\hat{B}_2]=\begin{bmatrix}\hat{B}_{11} & \hat{B}_{12}\\ \hat{B}_{21} & \hat{B}_{22}\end{bmatrix}, \tag{3.24}$$

$$F=[F_1|F_2] \tag{3.25}$$

where $\hat{B}_1$ and $\hat{B}_2$ are partitioned conformally with (3.11) and F is partitioned conformally with (3.9). Making use of (3.16), (3.23), (3.24) and (3.25), we can rewrite (2.24) as

$$\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & 0\end{bmatrix}(s)=\left[\begin{array}{cc|cc} J(A-B_2F)J^* & JB_2F & JB_1 & JB_2\\ 0 & A-HC_2 & -YC_2^* & 0\\ \hline -B_2^*J^*JXJ^* & F & D_{11} & I\\ 0 & C_2 & I & 0\end{array}\right]$$

$$=\left[\begin{array}{cccc|cc} -\Sigma_2^{-1}Z_{11}^*\Sigma_2 & 0 & \hat{B}_{12}F_1 & \hat{B}_{12}F_2 & \hat{B}_{11} & \hat{B}_{12}\\ Z_{21}-\hat{B}_{22}\hat{B}_{12}^*\Sigma_2 & Z_{22} & \hat{B}_{22}F_1 & \hat{B}_{22}F_2 & \hat{B}_{21} & \hat{B}_{22}\\ 0 & 0 & -\Sigma_1M_{11}^*\Sigma_1^{-1} & M_{12}-\Sigma_1C_{21}^*C_{22} & -\Sigma_1C_{21}^* & 0\\ 0 & 0 & 0 & M_{22} & 0 & 0\\ \hline -\hat{B}_{12}^*\Sigma_2 & 0 & F_1 & F_2 & D_{11} & I\\ 0 & 0 & C_{21} & C_{22} & I & 0\end{array}\right] \tag{3.26}$$

$$=\left[\begin{array}{cc|cc} -Z_{11}^* & \Sigma_2\hat{B}_{12}F_1\Sigma_1 & \Sigma_2\hat{B}_{11} & \Sigma_2\hat{B}_{12}\\ 0 & -M_{11}^* & -C_{21}^* & 0\\ \hline -\hat{B}_{12}^* & F_1\Sigma_1 & D_{11} & I\\ 0 & C_{21}\Sigma_1 & I & 0\end{array}\right].$$

Thus

$$T_{12}=\left[\begin{array}{c|c}-Z_{11}^* & \Sigma_2\hat{B}_{12}\\ \hline -\hat{B}_{12}^* & I\end{array}\right];\qquad T_{21}=\left[\begin{array}{c|c}-M_{11}^* & -C_{21}^*\\ \hline C_{21}\Sigma_1 & I\end{array}\right]. \tag{3.27}$$

It follows from (3.13) that $T_{21}(s)$ has observability gramian Σ_1 and controllability gramian Σ_1^{-1}. Similarly, $T_{12}(s)$ has observability gramian Σ_2^{-1} and controllability gramian Σ_2. Equations (3.18) and (3.22), $\Sigma_1>0$ and $\Sigma_2>0$ together with [12, Thm. 3.3(5)] establishes the minimality of the realizations in (3.27). To deduce that the realization (3.26) is also minimal, we note that the A-matrix of the inverse system of (3.26) is similar to

$$\begin{bmatrix} Z_{11} & (\hat{B}_{12}D_{11}-\hat{B}_{11})C_{21} \\ 0 & M_{11} \end{bmatrix}. \tag{3.28}$$

It follows from this that the realization (3.26) has no left half plane zeros and since this realization is asymptotically stable, it must be minimal as well because no pole-zero cancellations can occur. We remark however that the realization for $T_{11}(s)$ need not be minimal. Replacing the realization (2.24) by (3.26) allows the realization (2.28) for $T_{12}^*T_{11}T_{21}^*(s)$ to be replaced by

$$T_{12}^*T_{11}T_{21}^*(s)=\left[\begin{array}{cc|c} Z_{11} & (\hat{B}_{12}D_{11}-\hat{B}_{11})C_{21} & \hat{B}_{11}-\hat{B}_{12}D_{11} \\ 0 & M_{11} & -\Sigma_1 C_{21}^* \\ \hline -\hat{B}_{12}^*\Sigma_2 & F_1-D_{11}C_{21} & D_{11} \end{array}\right] \tag{3.29}$$

which need not be minimal. The results of our analysis up to this point are now summarized in the next lemma for easy reference.

LEMMA 3.1.

(i) *The number of zeros of* $P_{12}(s)$ *in* $\mathbb{C}_+=\operatorname{rank}(X)$.

(ii) *The number of zeros of* $P_{21}(s)$ *in* $\mathbb{C}_+=\operatorname{rank}(Y)$.

(iii) *The realization in* (3.26) *is minimal with degree* $=(\operatorname{rank}(X)+\operatorname{rank}(Y))$.

(iv) *The realizations for* $T_{12}(s)$ *and* $T_{21}(s)$ *in* (3.27) *are minimal with* $\deg(T_{12})=\operatorname{rank}(X)$ *and* $\deg(T_{21})=\operatorname{rank}(Y)$.

(v) $\deg(T_{12}^*T_{11}T_{21}^*)\leqq\operatorname{rank}(X)+\operatorname{rank}(Y)$.

Early in this section we showed that $\lambda(YX)$ are invariant with respect to an arbitrary similarity transformation in the state-space of $P(s)$. It is natural to ask whether or not these invariants contain any fundamental information pertaining to the optimal solutions of H^∞ control problems. We conclude this section with a result which shows that the lowest achievable L^∞-norm for the closed loop may be expressed in terms of $\lambda_{\max}(YX)$ in the case of certain specific problems of the first kind. These problems are: (i) The unweighted optimal sensitivity problem; (ii) the unweighted optimal complementary sensitivity problem; (iii) the unweighted problem associated with optimal robustness towards multiplicative perturbations at the plant input, and (iv) the weighted optimal robustness problem.

THEOREM 3.2. *If* Ξ *is the set of all stabilizing compensators, then*

$$\text{(i)}\quad \inf_{K\in\Xi}\|(I+GK)^{-1}(s)\|_\infty=(1+\lambda_{\max}(YX))^{1/2}; \tag{3.30}$$

$$\text{(ii)}\quad \inf_{K\in\Xi}\|GK(I+GK)^{-1}(s)\|_\infty=(1+\lambda_{\max}(YX))^{1/2}; \tag{3.31}$$

$$\text{(iii)}\quad \inf_{K\in\Xi}\|KG(I+KG)^{-1}(s)\|_\infty=(1+\lambda_{\max}(YX))^{1/2}; \tag{3.32}$$

(iv) *If* $W_1(s)$ *and* $W_2(s)$ *are stable and minimum phase frequency dependent weights with proper inverses, then*

$$\inf_{K\in\Xi}\|W_1K(I+GK)^{-1}W_2(s)\|_\infty=\lambda_{\max}(YX)^{1/2}. \tag{3.33}$$

□

Remark 3.1. A detailed analysis will reveal that the Riccati equations defining X and Y in the case of problems (i) and (ii) are the same and therefore that

$$\inf_{K\in\Xi}\|(I+GK)^{-1}\|_\infty=\inf_{K\in\Xi}\|GK(I+GK)^{-1}\|_\infty. \tag{3.34}$$

This result was originally proved by Kwakernaak in the SISO case and Glover in the MIMO case (private communication). In general, the (X, Y) pairs associated with the other problems given in Theorem 3.2 are different leading to different achievable L^∞-norm infima. It may be shown by counterexample that the results for problems (i), (ii) and (iii) do not carry over to the weighted case. □

4. Main results. In this section we consider the pole-zero cancellation properties of the H^∞-optimal (or suboptimal) system of Fig. 1 and we will derive general McMillan degree bounds for all H^∞-optimal controllers (denoted K_{opt}) or suboptimal controllers (denoted K_{sopt}) for problems of the first kind. An outline of our development is as follows.

Let $n=\deg(P)$, $t=\deg(\mathscr{R})$ and let $m=$(number of cancellations which occur between $P(s)$ and $K(s)$ as a result of closing the feedback loop in Fig. 1). Then

$$t=n+\deg(K)-m,$$

that is

$$\deg(K)=t+m-n.$$

To obtain an upper bound for $\deg(K)$, we proceed in two steps:

(1) Theorems 4.1, 4.2 and 2.1 establish an upper bound t_b for the McMillan degree t of all optimal closed-loop transfer functions $\mathscr{R}(s)$, and

(2) Theorem 4.3 establishes an upper bound m_b for the number of pole-zero cancellations between $P(s)$ and $K(s)$. Given such bounds, we then have

$$\deg(K)\leqq t_b+m_b-n. \tag{4.1}$$

In the case of single-input-single-output (SISO) problems we will show that

$$\deg(K_{\text{opt}})\leqq n-1, \tag{4.2}$$

$$\deg(K_{\text{sopt}})\leqq n. \tag{4.3}$$

In the case of multivariable (MIMO) problems, we will show that there is a continuum of controllers which satisfy the bounds given in (4.2) and (4.3). These results are stated in Theorem 4.4. We remark that a bound of this type has already been discovered by Glover in the special case of the optimal robustness problem [13].

In an earlier paper, Zames and Francis [28] established that there are interpolation constraints associated with both the right half plane poles and zeros of the plant in the single loop optimal sensitivity problem. If the weighted sensitivity is given by

$$s(s)=w(s)/(1+g(s)k(s)), \tag{4.4}$$

then they have shown that

$$s(z_i)=w(z_i) \tag{4.5}$$

at each right half plane zero z_i of $g(s)$, and that

$$s(p_i)=0 \tag{4.6}$$

at each right half plane pole. These observations lead us to an interesting factorization phenomenon which may occur in H^∞ control problems of the first kind. We will motivate the main idea by way of the unweighted sensitivity minimization problem (weights are neglected for ease of exposition).

From (2.6), we have after prescaling that

$$P(s)=\begin{bmatrix} I & G(s)D^{-1} \\ I & G(s)D^{-1} \end{bmatrix}=\left[\begin{array}{c|cc} A & 0 & BD^{-1} \\ \hline C & I & I \\ C & I & I \end{array}\right]. \tag{4.7}$$

Substituting into (2.20) to (2.24) we get

$$\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & 0 \end{bmatrix}=\left[\begin{array}{cc|cc} A-BD^{-1}F & YC^*C & YC^* & BD^{-1} \\ 0 & A-HC & -YC^* & 0 \\ \hline -D^{-*}B^*X & C & I & I \\ 0 & C & I & 0 \end{array}\right] \tag{4.8}$$

after the change of basis

$$T=\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}. \tag{4.9}$$

An inspection of (4.8) shows that it can be factorized as

$$\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & 0 \end{bmatrix}=\begin{bmatrix} \tilde{T}_{11}(s) & T_{12}(s) \\ I & 0 \end{bmatrix}\begin{bmatrix} T_{21}(s) & 0 \\ 0 & I \end{bmatrix} \tag{4.10}$$

where

$$\tilde{T}_{11}(s)=\left[\begin{array}{c|c} A-BD^{-1}F & YC^* \\ \hline -D^{-*}B^*X & I \end{array}\right]. \tag{4.11}$$

Consequently, $\mathcal{R}(s)$ can be written as

$$\mathcal{R}(s)=(\tilde{T}_{11}(s)-T_{12}Q(s))T_{21}(s) \tag{4.12}$$

in which we note also that {zeros of T_{21}} ≡ {poles of G} and {zeros of T_{12}} ≡ {zeros of G}. At each pole of $G(s)$ there exists a vector x_i such that

$$T_{21}(p_i)x_i=0 \quad \text{for all } Q(s) \text{ in } RH_+^{\infty} \tag{4.13}$$

which implies

$$\mathcal{R}(p_i)x_i=0. \tag{4.14}$$

This is a generalization of (4.6) to the MIMO case. The point we want to emphasize, however, is that in certain H^{∞} control problems, T_{11} may have natural all-pass common factors with T_{12} and/or T_{21}, as illustrated in (4.12). Theorem 4.1 gives a general treatment of the properties of this type of all-pass common factor.

THEOREM 4.1. *Let*

$$\begin{bmatrix} G & A_1 \\ A_2 & 0 \end{bmatrix}(s)=\left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} & B_{21} & 0 \\ \hline C_{11} & C_{12} & D & I \\ 0 & C_{22} & I & 0 \end{array}\right] \tag{4.15}$$

in which $A_1(s)$ and $A_2(s)$ are assumed inner. Suppose also that the realizations for $A_1(s)$ and $A_2(s)$ given in (4.15) are minimal. Then there exists a change of basis such that

(4.15) *can be put into the form*

$$\begin{bmatrix} G & A_1 \\ A_2 & 0 \end{bmatrix}(s) = \left[\begin{array}{cccc|cc} \tilde{A}_{00} & \tilde{A}_{01} & \tilde{A}_{02} & \tilde{A}_{03} & \tilde{B}_{01} & \tilde{B}_{02} \\ 0 & \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{B}_{11} & \tilde{B}_{12} \\ 0 & 0 & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{B}_{21} & 0 \\ 0 & 0 & 0 & \tilde{A}_{33} & \tilde{B}_{31} & 0 \\ \hline \tilde{C}_{10} & \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & D & I \\ 0 & 0 & \tilde{C}_{22} & \tilde{C}_{23} & I & 0 \end{array}\right], \tag{4.16}$$

which admits the following factorizations

$$A_1(s) = A_l(s)\tilde{A}_1(s) = \left[\begin{array}{c|c} \tilde{A}_{00} & \tilde{B}_{02} \\ \hline \tilde{C}_{10} & I \end{array}\right]\left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{B}_{12} \\ \hline \tilde{C}_{11} & I \end{array}\right], \tag{4.17}$$

$$A_2(s) = \tilde{A}_2(s)A_r(s) = \left[\begin{array}{c|c} \tilde{A}_{22} & \tilde{B}_{21} \\ \hline \tilde{C}_{22} & I \end{array}\right]\left[\begin{array}{c|c} \tilde{A}_{33} & \tilde{B}_{31} \\ \hline \tilde{C}_{23} & I \end{array}\right], \tag{4.18}$$

$$G(s) = A_l(s)\tilde{G}(s)A_r(s) = \left[\begin{array}{c|c} \tilde{A}_{00} & \tilde{B}_{02} \\ \hline \tilde{C}_{10} & I \end{array}\right]\left[\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_{11} \\ 0 & \tilde{A}_{22} & \tilde{B}_{21} \\ \hline \tilde{C}_{11} & \tilde{C}_{12} & D \end{array}\right]\left[\begin{array}{c|c} \tilde{A}_{33} & \tilde{B}_{31} \\ \hline \tilde{C}_{23} & I \end{array}\right]. \tag{4.19}$$

Further

(a) $\tilde{A}_1(s)$, $\tilde{A}_2(s)$, $A_l(s)$, $A_r(s)$, *are inner.*

(b) *The factorizations in* (4.17), (4.18), *and* (4.19) *are minimal in the sense that*

$$\deg(A_1) = \deg(A_l) + \deg(\tilde{A}_1), \tag{4.20}$$

$$\deg(A_2) = \deg(A_r) + \deg(\tilde{A}_2), \tag{4.21}$$

$$\deg(G) = \deg(A_l) + \deg(\tilde{G}) + \deg(A_r). \tag{4.22}$$

(c) $A_1^*GA_2^*(s) = \tilde{A}_1^*\tilde{G}\tilde{A}_2^*(s)$ *has a minimal realization given by*

$$\left[\begin{array}{cc|c} -\tilde{A}_{11}^* & \tilde{A}_{12} + \tilde{B}_{11}\tilde{B}_{21}^* + \tilde{C}_{11}^*(\tilde{C}_{12} + D\tilde{B}_{21}^*) & \tilde{C}_{11}^*D + \tilde{B}_{11} \\ 0 & -\tilde{A}_{22}^* & -\tilde{C}_{22}^* \\ \hline -\tilde{B}_{12}^* & \tilde{C}_{12} + D\tilde{B}_{21} & D \end{array}\right]. \tag{4.23}$$ □

The proof of this result, which is inspired by the work of Van Dooren and DeWilde [23], is given in Appendix C.

Theorem 2.1 shows that the realization (3.29) for $T_{12}^*T_{11}T_{21}^*(s)$ is controllable if and only if $T_{11}(s)$ and $T_{12}(s)$ have no common inner left divisors and that it is observable if and only if $T_{11}(s)$ and $T_{21}(s)$ have no common inner right divisors. The realization is thus minimal if and only if neither type of factor exists. However, if such inner common factors do exist, they may be extracted to form the cascade factorization

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix}(s) = \begin{bmatrix} A_l(s) & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & 0 \end{bmatrix}(s)\begin{bmatrix} A_r(s) & 0 \\ 0 & I \end{bmatrix} \tag{4.24}$$

in which $A_l(s)$ and $A_r(s)$ are inner. Consequently, we have

$$\mathcal{R}(s) = A_l(s)[\tilde{T}_{11} - \tilde{T}_{12}Q\tilde{T}_{21}](s)A_r(s). \tag{4.25}$$

Furthermore, a minimal realization for

$$\begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & 0 \end{bmatrix}(s) \tag{4.26}$$

will lead directly to a minimal realization for $\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*(s)$. Hence the function to be minimized in (2.25) or (2.26) can be written

$$\|(T_{11}-T_{12}QT_{21})(s)\|_\infty=\|(\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*-Q)(s)\|_\infty. \tag{4.27}$$

Using Theorem 2.1 or Remark 2.2, $Q(s)$ can therefore be obtained as a Nehari extension or ρ-suboptimal extension of $\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*(s)$. It then follows from Theorem 2.1 and Remarks 2.2 and 2.4 that the corresponding "error system"

$$E(s)=(\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*-Q)(s) \tag{4.28}$$

satisfies the bounded real type equations given in (2.33) and (2.34). These equations form the basis of the hypothesis of the next theorem which enables us to deduce that the set of poles of $(\tilde{T}_{11}-\tilde{T}_{12}Q\tilde{T}_{21})(s)$ reduce to a subset of the poles of Q.

THEOREM 4.2. *Let*

$$\begin{bmatrix} G & A_1 \\ A_2 & 0 \end{bmatrix}(s)=\left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} & B_{21} & 0 \\ \hline C_{11} & C_{12} & D & I \\ 0 & C_{22} & I & 0 \end{array}\right] \tag{4.29}$$

in which $A_1(s)$ and $A_2(s)$ are inner and their realizations

$$A_1(s)=\left[\begin{array}{c|c} A_{11} & B_{12} \\ \hline C_{11} & I \end{array}\right] \quad \text{and} \quad A_2(s)=\left[\begin{array}{c|c} A_{22} & B_{21} \\ \hline C_{22} & I \end{array}\right] \tag{4.30}$$

are also minimal and balanced. Then

(a)

$$\begin{aligned} A_1^*GA_2^*(s)&=\left[\begin{array}{cc|c} -A_{11}^* & -C_{11}^*(C_{12}+DB_{21}^*)-A_{12}-B_{11}B_{21}^* & C_{11}^*D+B_{11} \\ 0 & -A_{22}^* & C_{22}^* \\ \hline -B_{12}^* & -(C_{12}+DB_{21}^*) & D \end{array}\right] \\ &=\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{bmatrix}; \end{aligned} \tag{4.31}$$

(b) *for any*

$$Q(s)=\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

such that

$$(A_1^*GA_2^*-Q)(s)=\left[\begin{array}{cc|c} \underline{A} & 0 & \underline{B} \\ 0 & \hat{A} & \hat{B} \\ \hline \underline{C} & -\hat{C} & \underline{D}-\hat{D} \end{array}\right]:=\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] \tag{4.32}$$

satisfies the bounded real type equations

$$\begin{bmatrix} -(\tilde{A}\tilde{P}+\tilde{P}\tilde{A}^*+\tilde{B}\tilde{B}^*) & -(\tilde{B}\tilde{D}^*+\tilde{P}\tilde{C}^*) \\ -(\tilde{D}\tilde{B}^*+\tilde{C}\tilde{P}) & \sigma^2I-\tilde{D}\tilde{D}^* \end{bmatrix}=\begin{bmatrix} L \\ W \end{bmatrix}[L^*|W^*] \tag{4.33}$$

and

$$\begin{bmatrix} -(\tilde{A}^*\tilde{Q}+\tilde{Q}\tilde{A}+\tilde{C}^*\tilde{C}) & -(\tilde{C}^*\tilde{D}+\tilde{Q}\tilde{B}) \\ -(\tilde{D}^*\tilde{C}+\tilde{B}^*\tilde{Q}) & \sigma^2I-\tilde{D}^*\tilde{D} \end{bmatrix}=\begin{bmatrix} L_d^* \\ W_d^* \end{bmatrix}[L_d|W_d] \tag{4.34}$$

in which

(4.35) (i) $\tilde{P}\tilde{Q}=\sigma^2 I$ *for some* $\sigma\in\mathbb{R}$, *and*

(4.36) (ii) $L^*=[0|L_{21}^*]$, $L_d=[0|L_{d21}]$

where the partitioning of L *and* L_d *is conformable with that of* $\tilde{B}$ *and* $\tilde{C}$ *in* (4.32), *we have*

$$(4.37)\qquad \text{(i)}\quad (G-A_1QA_2)(s)=\left[\begin{array}{c|c}\hat{A} & \hat{B}-P_{23}^*B_{21}\\ \hline C_{11}Q_{13}-\hat{C} & \tilde{D}\end{array}\right]$$

where P_{23} *and* Q_{13} *are the* (2, 3) *and* (1, 3) *partitions of* $\tilde{P}$ *and* $\tilde{Q}$ (*see* (D.1) *in Appendix* D).

(4.38) (ii) $\{$*poles of* $Q\}\supseteq\{$*poles of* $(G-A_1QA_2)\}$.

Proof. See Appendix D.

Clearly, if we substitute $\tilde{T}_{11}(s)$, $\tilde{T}_{12}(s)$ and $\tilde{T}_{21}(s)$ into $G(s)$, $A_1(s)$, $A_2(s)$ of the last theorem, it follows immediately from part (b)(ii) that

$$(4.39)\qquad \{\text{poles of } Q\}\supseteq\{\text{poles }(\tilde{T}_{11}-\tilde{T}_{12}Q\tilde{T}_{21})\}$$

and this together with Theorem 4.1 and (4.25) yields

$$(4.40)\qquad \{\text{poles of } Q\}\cup\{\text{poles of } A_r\}\cup\{\text{poles of } A_l\}\supseteq\{\text{poles of } \mathcal{R}\}.$$

Given (4.40), it follows that an upper bound for the McMillan degree of $\mathcal{R}(s)$ is

$$(4.41)\qquad t\leqq \deg(A_r)+\deg(A_l)+\deg(Q)=t_b.$$

Further, we have by Corollary 2.2 that

$$(4.42)\qquad \deg(Q_{\text{opt}})\leqq \deg(\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*)+\deg(U)-r$$

in the case of optimal extensions (r is the multiplicity of the largest Hankel singular value of $\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*(s)$); and

$$(4.43)\qquad \deg(Q_{\text{sopt}})\leqq \deg(\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*)+\deg(U)$$

in the case of ρ-suboptimal extensions. Now (3.26) and Lemma 3.1 in combination with (4.24) and Theorem 4.1 imply that

$$(4.44)\qquad \deg(\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*)=\text{rank}(X)+\text{rank}(Y)-\deg(A_r)-\deg(A_l).$$

Direct substitution of this into the previous two inequalities and then into (4.41) yields

$$(4.45\text{a})\qquad t_{\text{opt}}\leqq t_b=\text{rank}(X)+\text{rank}(Y)+\deg(U)-r$$

and

$$(4.45\text{b})\qquad t_{\text{sopt}}\leqq t_b=\text{rank}(X)+\text{rank}(Y)+\deg(U)$$

which provides a McMillan degree bound for the closed-loop system and completes step 1 of our analysis.

We will now begin the second step. In order to establish a McMillan degree bound on all $K_{\text{opt}}(s)$ and $K_{\text{sopt}}(s)$ controllers, we need to bound the number of cancellations between $P(s)$ and $K(s)$ in Fig. 1; we call this bound m_b as was given in (4.1). In Theorem 4.3, we will show that every unobservable mode of the system in Fig. 1 is due to a cancellation with a zero of $P_{12}(s)$ and every uncontrollable mode is due to a cancellation with a zero of $P_{21}(s)$. (After this paper had been submitted for publication, it came to our attention that a result similar to Theorem 4.3 had been discovered independently by Anderson and Linnemann [29].)

THEOREM 4.3. *Let*

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{4.46}$$

in which $P_{12}(s) \in \mathbb{R}^{p_1 \times m_2}(s)$ *with* $m_2 \leqq p_1$ *and* $P_{21}(s) \in \mathbb{R}^{p_2 \times m_1}(s)$ *with* $p_2 \leqq m_1$. *Suppose also that*

$$K(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \tag{4.47}$$

is a minimal realization and that the well posedness condition $\det(I + D_{22}\hat{D}) \neq 0$ *is satisfied. Then in the closed loop of Fig.* 1

(a) *every unobservable mode is a zero of* $P_{12}(s)$ *and*

(b) *every uncontrollable mode is a zero of* $P_{21}(s)$.

Proof. See Appendix E.

Using this theorem, we see that the number of cancellations m between $P(s)$ and $K(s)$ is bounded above by

$$\begin{aligned} m \leqq &\{\text{number of zeros of } P_{12}(s) \text{ in } \mathbb{C}_-\} \\ &+ \{\text{number of zeros of } P_{21}(s) \text{ in } \mathbb{C}_-\} = m_b. \end{aligned} \tag{4.48}$$

This follows from the fact that any other cancellation (i.e. one corresponding to a right half plane zero of $P_{12}(s)$ or $P_{21}(s)$) violates the proven internal stability of the closed loop.

From (4.48) and Lemma 3.1, we have

$$\begin{aligned} m_b &= \{\text{number of zeros of } P_{12}(s) \text{ in } \mathbb{C}_-\} \\ &\quad + \{\text{number of zeros of } P_{21}(s) \text{ in } \mathbb{C}_-\} \\ &= \{n - \operatorname{rank}(X)\} + \{n - \operatorname{rank}(Y)\} \\ &= 2n - \operatorname{rank}(X) - \operatorname{rank}(Y). \end{aligned} \tag{4.49}$$

We are now ready to prove the main theorem by combining together the results which have been established. Substitution of (4.45) and (4.49) into (4.1) proves the following.

THEOREM 4.4. *For any H^∞-optimal control problem of the first kind, every H^∞-optimal controller satisfies*

(i) $\deg(K_{\text{opt}}) \leqq n - r + \deg(U)$

where r is the multiplicity of the largest Hankel singular value of $\tilde{T}_{12}^* \tilde{T}_{11} \tilde{T}_{21}^*(s)$, *and every ρ-suboptimal controller satisfies*

(ii) $\deg(K_{\text{sopt}}) \leqq n + \deg(U)$. □

Furthermore, (4.2) and (4.3) follow from the fact that $\deg(U) = 0$ if $U(s)$ is chosen constant. In the SISO case only $U = 1$ is allowed.

4.1. Computations. In this section we assemble together the ideas presented so far into an algorithm style procedure for solving H^∞ control problems of the first kind.

ALGORITHM.

(1) Given $P(s)$ as in (2.3), do a prescaling to get $P(s)$ into the form (2.19).

(2) Solve the Riccati equations (2.21) and (2.23) and evaluate the stabilizing matrices F and H using (2.20) and (2.22).

(3) Assemble $T_{12}^* T_{11} T_{21}^*(s)$ as in (2.28).

(4) Remove the hidden modes of $T_{12}^* T_{11} T_{21}^*(s)$. This can be done in a numerically reliable way by balanced truncation methods. The result is a minimal realization for $\tilde{T}_{12}^* \tilde{T}_{11} \tilde{T}_{21}^*(s)$. Note the balancing process for model reduction at this step forms the bulk of the computation required at the next step.

(5) Determine a Nehari or ρ-suboptimal extension $Q(s)$ of $\tilde{T}_{12}^* \tilde{T}_{11} \tilde{T}_{21}^*(s)$ using Theorem 2.1 or Remark 2.2.

(6) Back substitute $Q(s)$ into (2.17).

(7) The previous step will typically produce a nonminimal realization for the (sub-)optimal controller $K(s)$. Again, a minimal realization may be obtained by balanced truncation methods; the bounds given in Theorem 4.4 must apply.

It should be noted that the model reduction performed in step 4 will simultaneously remove all the nonminimal states in (2.28) introduced by the left half plane zeros of $P_{12}(s)$ and $P_{21}(s)$, and the all-pass common factors shared by $T_{11}(s)$ and $T_{12}(s)$, and $T_{11}(s)$ and $T_{21}(s)$. Although Theorem 4.1 is an essential component of the theory, it does not need to be implemented in software. The second model reduction (step 7) is used to remove any nonminimal states introduced by cancellations predicted by Theorem 4.3.

4.2. Model reduction considerations. It is natural to consider reducing the number of controller states by model reduction methods such as those discussed in [12]. If we suppose that $\Delta\mathscr{R}(s)$ is the change in $\mathscr{R}(s)$ produced by the model reduction error $\Delta K(s)$, then the difficulty with this approach is that any general bound on $\|\Delta\mathscr{R}(s)\|_\infty$ in terms of a bound on $\|\Delta K(s)\|_\infty$ tends to be weak. An alternative and less direct approach is to consider the possibility of model reducing $Q(s)$ before obtaining $K(s)$ by back substitution. An argument might be that if $\Delta Q(s)$ is the perturbation produced by the model reduction of $Q(s)$, then by (2.14)

$$\mathscr{R}(s)+\Delta\mathscr{R}(s)=[T_{11}-T_{12}(Q+\Delta Q)T_{21}](s) \tag{4.50}$$

leads to

$$\|\Delta\mathscr{R}(s)\|_\infty=\|\Delta Q(s)\|_\infty, \tag{4.51}$$

since $T_{12}(s)$ and $T_{21}(s)$ are inner. Further, if the reduced order model of $Q(s)$ is obtained by retaining the first k states of a truncated balanced realization, then

$$\|\Delta\mathscr{R}(s)\|_\infty \leqq 2 \sum_{i=k+1}^{n} \sigma_i(Q). \tag{4.52}$$

This inequality follows from [12, Thm. 9.6] and shows that it is possible to reduce the number of states of $Q(s)$ while simultaneously keeping track of the resulting maximum possible increase in $\|\mathscr{R}(s)\|_\infty$. However, contrary to the objective, this approach will tend to increase the number of controller states rather than decrease it. This is because replacing $Q(s)$ with a lower order approximation will destroy the "built in" cancellations predicted by our previous results. Since $K(s)=F_l(K_0(s), Q(s))$ and deg $(K_0)=n$

we see that

$$
\begin{aligned}
\deg(K_{\text{opt}}) &= \deg(K_0) + \deg(Q) \\
&\quad -(\text{no. of cancellations between } K_0 \text{ and } Q_{\text{opt}}) \qquad (4.53)\\
&\leqq n-1 \ (\text{in the case } \deg(U)=0 \text{ in Theorem 4.4}).
\end{aligned}
$$

If Q_{opt} is replaced by an approximation $Q_a(s)$, then in general the cancellations no longer occur and the corresponding controller $K_a(s)$ has higher degree than that of $K_{\text{opt}}(s)$, specifically

$$\deg(K_a) = \deg(K_0) + \deg(Q_a) \geqq n.$$

This point is now illustrated with an example.

Example 4.1. Consider the unweighted robust stabilization problem in which we seek

$$\inf_{K \in \Xi} \|K(I+GK)^{-1}\|_\infty \quad (\Xi \text{ is the set of stabilizing compensators}).$$

Referring back to (2.8) we recall that the corresponding $P(s)$ matrix is

$$P(s) = \begin{bmatrix} 0 & I \\ -I & G(s) \end{bmatrix}$$

and after scaling ($S_2 = -I$ and $S_1 = I$) we get

$$P(s) = \left[\begin{array}{c|cc} A & 0 & B \\ \hline 0 & 0 & I \\ -C & I & -D \end{array}\right].$$

If

$$G(s) = \{(s+3)\}/\{(s-1)(s-2)(s-3)\},$$

we get (by computer) a calculation which is based on § 4.1 that

$$Q_{\text{opt}} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} -1.54049 & 0.89744 & 307.989 \\ -0.912165 & -0.17001 & 60.1637 \\ \hline -0.083009 & 0.0159534 & 61.4750 \end{array}\right]$$

which has Hankel singular values 8.2979 and 2.8229. The corresponding optimal controller $K_{\text{opt}}(s)$ has McMillan degree two and has Hankel singular values 38.084 and 8.3797.

In a second calculation, we replaced $Q_{\text{opt}}(s)$ with a 1-state truncated balanced realization. In this case $K(s)$ had McMillan degree four with Hankel singular values: 38.065, 8.3596, 0.0016269 and 0.00010428. In this example therefore, removing a state from $Q_{\text{opt}}(s)$ leads to an increase of two in the McMillan degree of the controller. □

4.3. Minimum entropy controllers. In a private communication N. J. Young pointed out to us that Arov and Krein [2] had studied a class of ρ-suboptimal extensions, which they called minimum entropy extensions. Assuming that $\|\mathcal{R}(s)\|_\infty \leqq \rho = 1$ (a convenient normalization), we define the entropy of the closed loop system at some point $s_0 \in \mathbb{C}_-$ by

$$I(\mathcal{R}; s_0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln|\det(I - \mathcal{R}^*(j\omega)\mathcal{R}(j\omega))| \frac{\operatorname{Re}(s_0)}{|j\omega - s_0|^2}\, d\omega. \qquad (4.54)$$

Using the fact that $\det(I+AB)=\det(I+BA)$, it is easy to see that inner matrices are entropy preserving. Since $\mathcal{R}(s)=T_{12}ET_{21}(s)$, where $E(s)$ is given by (4.28), it is clear that $I(\mathcal{R}; s_0)=I(E; s_0)$. From now on we will work with $E(s)$ knowing that it has the same entropy as the closed loop transfer function matrix $\mathcal{R}(s)$.

After some introductory comments we will state the result of Arov and Krein and thus show that (4.54) may be minimized while ensuring $\|\mathcal{R}(s)\|_\infty \leqq 1$ with the aid of an *n-state controller.* It will be shown that the controller which minimizes the closed loop entropy is generated by setting $U=-H_{22}^*(s_0)$ in the general parametrization of all Nehari extensions.

From [12] we recall that

$$(4.55)\qquad \begin{aligned} E^*(s) &= (\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*)^*(s)-H_{11}(s)\\ &\quad +H_{12}(s)U(s)(I+H_{22}(s)U(s))^{-1}H_{21}(s), \end{aligned}$$

or alternatively,

$$(4.56)\qquad E^*(s)=\{A(s)U(s)+B(s)\}\{C(s)U(s)+D(s)\}^{-1}$$

in which

$$(4.57)\qquad A(s):=\{(\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*)^*(s)-H_{11}\}H_{21}^{-1}(s)H_{22}(s)+H_{12}(s),$$

$$(4.58)\qquad B(s):=\{(\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*)^*(s)-H_{11}\}H_{21}^{-1}(s),$$

$$(4.59)\qquad C(s):=H_{21}^{-1}(s)H_{22}(s),$$

$$(4.60)\qquad D(s):=H_{21}^{-1}(s).$$

Substituting from (2.38), using $\rho=1$, we get

$$(4.61)\qquad \begin{bmatrix} A(s) & B(s)\\ C(s) & D(s)\end{bmatrix}=\left[\begin{array}{cc|cc} A & 0 & -\Sigma\Gamma^{-1}C^* & -\Gamma^{-1}B\\ 0 & -A^* & \Gamma^{-1}C^* & \Gamma^{-1}\Sigma B\\ \hline C & 0 & I & 0\\ 0 & B^* & 0 & I\end{array}\right]$$

in which we have denoted

$$(\tilde{T}_{12}^*\tilde{T}_{11}\tilde{T}_{21}^*)^*(s)=\left[\begin{array}{c|c} A & B\\ \hline C & D\end{array}\right].$$

Also, (4.61) is easily shown to be J-unitary, that is

$$(4.62)\qquad \begin{bmatrix} A(s) & B(s)\\ C(s) & D(s)\end{bmatrix}\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix}\begin{bmatrix} A^*(s) & C^*(s)\\ B^*(s) & D^*(s)\end{bmatrix}=\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix}.$$

From the J-unitary equations we get

$$(4.63)\qquad A(s)A^*(s)-B(s)B^*(s)=I,$$

$$(4.64)\qquad C(s)C^*(s)-D(s)D^*(s)=-I,$$

$$(4.65)\qquad A(s)C^*(s)-B(s)D^*(s)=0.$$

Also

$$(4.66)\qquad (4.64)\Rightarrow D^{-1}(s)D^{-*}(s)=I-H_{22}(s)H_{22}^*(s)$$

$$(4.67)\qquad \Rightarrow \|H_{22}(s)\|_\infty \leqq 1.$$

Glover [12] has shown that $H_{22}(s)\in RH_-^\infty$; a fact which is required in the proof of Theorem 4.5.

THEOREM 4.5 (Arov and Krein [2]). *If* $I(H_{22}; s_0) < \infty$, $s_0 \in \mathbb{C}_-$ *and* $\|U(s)\|_\infty \leq 1$, *then*

$$I(\mathcal{R}; s_0) = I(H_{22}; s_0) + I(U; s_0) + \ln |\det (I + H_{22}(s_0) U(s_0))|.$$

Also, there exists a unique U_0 *such that* $I(\mathcal{R}; s_0)$ *attains a minimum value of*

$$I(\mathcal{R}; s_0) = I(H_{22}; s_0) + \ln |\det (I - H_{22}(s_0) H_{22}^*(s_0))|/2.$$

The optimizing U_0 *is given by*

$$U_0 = -H_{22}^*(s_0).$$

A proof which mimics the discrete time proof in [2] is given in Appendix F.

5. Conclusions. Our purpose was to carry out a detailed analysis of the pole-zero cancellations which occur in the class of H^∞-optimal control problems described in § 2.2. If $\deg (P) = n$, we have shown that SISO H^∞ controllers never require more than $n-1$ states and that MIMO problems have a continuum of controllers whose McMillan degree satisfy this same bound. A general bound on $\deg (K)$ has been derived for all Nehari and ρ-suboptimal extensions and is given in Theorem 4.4. The bounds in Theorem 4.4 are tight in the sense that there exist problems for which they are met with equality. We have found in numerous examples that these bounds typically give the actual McMillan degree of the controller.

It is our belief that state-space dimension inflation is an important consideration in practical H^∞ design problems. Apart from being interesting in its own right, a complete cancellation theory is a prerequisite for the development of reliable computational software. Example 4.1 is an illustration of how a seemingly sensible, but ill-advised intermediate model reduction step may aggravate the problem of degree inflation rather than alleviate it.

H^∞ design problems which may be embedded in Fig. 1 but with either $P_{12}(s)$ or $P_{21}(s)$, or both, nonsquare have been studied by several researchers [6], [9], [10], [11], [22], [25]. In this class of problems cancellation phenomena are more difficult to analyse [16]. However, the added complexity and iterative nature of their solution makes a cancellation theory even more essential. An additional layer of difficulty is introduced by the various scaling strategies which are introduced in the μ-synthesis work of Doyle [5], Doyle et al. [6], Safonov [20] and others.

Appendix A.

Proof of Theorem 2.1. By assumption, the Hankel singular values have been ordered so that

$$\text{(A.1)}\quad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \sigma_{k+1} I \end{bmatrix} + \begin{bmatrix} \Sigma & 0 \\ 0 & \sigma_{k+1} I \end{bmatrix} \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} [B_1^* | B_2^*] = 0$$

and

$$\text{(A.2)}\quad \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \sigma_{k+1} I \end{bmatrix} + \begin{bmatrix} \Sigma & 0 \\ 0 & \sigma_{k+1} I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} [C_1 | C_2] = 0$$

are satisfied.

Next, we invoke [12, Thm. 8.7] which states that all error systems with the desired properties may be generated by

$$\text{(A.3)}\quad \hat{X}(s) + Q^*(s) = H_{11}(s) - H_{12}(s) U(s) (I + H_{22}(s) U(s))^{-1} H_{21}(s)$$

in which

$$H(s)=\begin{bmatrix} H_{11}(s)H_{12}(s) \\ H_{21}(s)H_{22}(s) \end{bmatrix}=\left[\begin{array}{c|cc} \Gamma^{-1}(\sigma_{k+1}^2 A_{11}^*+\Sigma A_{11}\Sigma) & \Gamma^{-1}\Sigma B_1 & -\Gamma^{-1}C_1^* \\ \hline C_1\Sigma & D & I \\ -\sigma_{k+1}B_1^* & \sigma_{k+1}I & 0 \end{array}\right] \tag{A.4}$$

and

$$U(s)\in RH_-^\infty \tag{A.5}$$

satisfies

$$\|U(j\omega)\|_\infty \leqq 1 \tag{A.6}$$

and

$$C_2+U(s)B_2^*=0. \tag{A.7}$$

In order that we may obtain the required state-space characterization of all error systems we assume that an arbitrary $U(s)$ with the desired properties has a minimal state-space realization $U(s)=(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. Since $\|U(j\omega)\|_\infty \leqq 1 \Leftrightarrow I-U(j\omega)U^*(j\omega)\geqq 0$, the bounded real lemma [1, p. 308] ensures the existence of $\hat{P}=\hat{P}^*<0$, $\hat{Q}=\hat{Q}^*<0$, $\hat{L}$, $\hat{W}$, $\hat{L}_d$ and $\hat{W}_d$ such that

$$\begin{bmatrix} -(\hat{A}\hat{P}+\hat{P}\hat{A}^*+\hat{B}\hat{B}^*) & -(\hat{B}\hat{D}^*+\hat{P}\hat{C}^*) \\ -(\hat{D}\hat{B}^*+\hat{C}\hat{P}) & I-\hat{D}\hat{D}^* \end{bmatrix}=\begin{bmatrix} \hat{L} \\ \hat{W} \end{bmatrix}[\hat{L}^*|\hat{W}^*] \tag{A.8}$$

and

$$\begin{bmatrix} -(\hat{A}^*\hat{Q}+\hat{Q}\hat{A}+\hat{C}^*\hat{C}) & -(\hat{C}^*\hat{D}+\hat{Q}\hat{B}) \\ -(\hat{D}^*\hat{C}+\hat{B}^*\hat{Q}) & I-\hat{D}^*\hat{D} \end{bmatrix}=\begin{bmatrix} \hat{L}_d^* \\ \hat{W}_d^* \end{bmatrix}[\hat{L}_d|\hat{W}_d] \tag{A.9}$$

are satisfied.

$$(\text{A.7}) \Rightarrow C_2+U(\infty)B_2^*=0 \Rightarrow C_2+\hat{D}B_2^*=0, \tag{A.10}$$

$$\begin{aligned} (\text{A.7}) \text{ and } (\text{A.10}) &\Rightarrow \hat{C}(sI-\hat{A})^{-1}\hat{B}B_2^*=0 \\ &\Rightarrow \hat{B}B_2^*=0 \quad (\text{since } [\hat{A}, \hat{C}] \text{ is observable}). \end{aligned} \tag{A.11}$$

The condition $\|U(j\omega)\|_\infty \leqq 1$ implies that there exists a spectral factor $\Delta(s)$ such that

$$I-U^*(s)U(s)=\Delta^*(s)\Delta(s), \tag{A.12}$$

$$(\text{A.7}) \Rightarrow B_2U^*(s)U(s)B_2^*=C_2^*C_2,$$

$$(\text{A.12}) \Rightarrow B_2B_2^*-B_2\Delta^*(s)\Delta(s)B_2^*=C_2^*C_2. \tag{A.13}$$

The (2, 2) blocks of (A.1) and (A.2) $\Rightarrow B_2B_2^*=C_2^*C_2$ and this together with (A.13) means

$$\Delta(s)B_2^*=0. \tag{A.14}$$

Multiplying (A.7) on the left by $U^*(s)$ we get

$$\begin{aligned} &U^*(s)C_2+U^*(s)U(s)B_2^*=0 \\ &\Leftrightarrow U^*(s)C_2+B_2^*-\Delta^*(s)\Delta(s)B_2^*=0 \quad (\text{by } (\text{A.12})) \\ &\Rightarrow U^*(s)C_2+B_2^*=0 \quad (\text{by } (\text{A.14})). \end{aligned} \tag{A.15}$$

Hence

$$\text{(A.16)} \qquad \text{(A.15)} \Rightarrow U^*(\infty)C_2 + B_2^* = \hat{D}^*C_2 + B_2^* = 0,$$

$$\text{(A.17)} \qquad \text{(A.15) and (A.16)} \Rightarrow \hat{B}^*(sI + \hat{A}^*)^{-1}\hat{C}^*C_2 = 0$$

$$\text{(A.18)} \qquad \Rightarrow \hat{C}^*C_2 = 0 \quad (\text{by the controllability of } [\hat{A}, \hat{B}]).$$

The state-space model (2.32) for $E(s)$ can be obtained by deriving a state-space realization for the linear fractional transformation (A.3) from the realizations of $H(s)$ and $U(s)$.

Equations (2.33) and (2.34) are proved by simple calculations which are reminiscent of those in [12]. We will begin with the (1, 1) block of (2.33). The validity of partitions (1, 1), (1, 2), (2, 1) and (2, 2) follows directly from (A.1).

$$\begin{aligned}
\text{partition } (1,3) &= A_{11} + (\sigma_{k+1}^2 A_{11} + \Sigma A_{11}^*\Sigma - \sigma_{k+1}B_1\hat{D}^*C_1)\Gamma^{-1} \\
&\quad + B_1(B_1^*\Sigma + \sigma_{k+1}\hat{D}^*C_1)\Gamma^{-1} \\
&= (A_{11}\Sigma^2 - \sigma_{k+1}^2 A_{11} + \sigma_{k+1}^2 A_{11} + \Sigma A_{11}^*\Sigma \\
&\qquad - (A_{11}\Sigma + \Sigma A_{11}^*)\Sigma)\Gamma^{-1} \quad (\text{by (A.1)}) \\
&= 0.
\end{aligned}$$

$$\text{partition } (1,4) = -\sigma_{k+1}B_1\hat{B}^* + \sigma_{k+1}B_1\hat{B}^* = 0.$$

$$\begin{aligned}
\text{partition } (2,3) &= A_{21} + B_2(B_1^*\Sigma + \sigma_{k+1}\hat{D}^*C_1)\Gamma^{-1} \\
&= (A_{21}\Sigma^2 - \sigma_{k+1}^2 A_{21} - (\sigma_{k+1}A_{12}^* + A_{21}\Sigma)\Sigma \\
&\quad + (A_{12}^*\Sigma + \sigma_{k+1}A_{21})\sigma_{k+1})\Gamma^{-1} \quad (\text{by (A.1, A.2, A.10)}) \\
&= 0.
\end{aligned}$$

$$\text{partition } (2,4) = \sigma_{k+1}B_2\hat{B}^* = 0 \quad (\text{by (A.11)}).$$

$$\begin{aligned}
\text{partition } (3,1) &= \Gamma^{-1}\{\sigma_{k+1}^2 A_{11}^* + \Sigma A_{11}\Sigma - \sigma_{k+1}C_1^*\hat{D}B_1^* + \Sigma^2 A_{11}^* - \sigma_{k+1}^2 A_{11}^* \\
&\qquad + (\Sigma B_1 + \sigma_{k+1}C_1^*\hat{D})B_1^*\} \\
&= \Gamma^{-1}\{\Sigma A_{11}\Sigma + \Sigma^2 A_{11}^* - \Sigma(A_{11}\Sigma + \Sigma A_{11}^*)\} \quad (\text{by (A.1)}) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\text{partition } (3,2) &= \Gamma^{-1}\{\Sigma^2 A_{21}^* - \sigma_{k+1}^2 A_{21}^* + (\Sigma B_1 + \sigma_{k+1}C_1^*\hat{D})B_2^*\} \\
&= \Gamma^{-1}\{\Sigma^2 A_{21}^* - \sigma_{k+1}^2 A_{21}^* - \Sigma(\sigma_{k+1}A_{12} + \Sigma A_{21}^*) \\
&\qquad + \sigma_{k+1}(\sigma_{k+1}A_{12}^* + \Sigma A_{12})\} \\
&= 0 \quad (\text{by (A.1), (A.2), (A.10)}).
\end{aligned}$$

$$\begin{aligned}
\text{partition } (3,3) &= \Gamma^{-1}\{(\sigma_{k+1}^2 A_{11}^* + \Sigma A_{11}\Sigma - \sigma_{k+1}C_1^*\hat{D}B_1^*)\Sigma \\
&\quad + \Sigma(\sigma_{k+1}^2 A_{11} + \Sigma A_{11}^*\Sigma - \sigma_{k+1}B_1\hat{D}^*C_1) \\
&\quad + (\Sigma B_1 + \sigma_{k+1}C_1^*\hat{D})(B_1^*\Sigma + \sigma_{k+1}\hat{D}^*C_1) \\
&\qquad + \sigma_{k+1}^2 C_1^*\hat{W}\hat{W}^*C_1\}\Gamma^{-1} \\
&= \Gamma^{-1}\{\sigma_{k+1}^2 A_{11}^*\Sigma + \Sigma A_{11}\Sigma^2 + \sigma_{k+1}^2\Sigma A_{11}^* + \Sigma^2 A_{11}\Sigma \\
&\quad - \Sigma(A_{11}\Sigma + \Sigma A_{11}^*)\Sigma \\
&\qquad - \sigma_{k+1}^2(A_{11}^*\Sigma + \Sigma A_{11})\}\Gamma^{-1} \\
&= 0 \quad (\text{by (A.1), (A.2), (A.8)}).
\end{aligned}$$

$$\begin{aligned}
\text{partition } (3,4) &= \Gamma^{-1}\{\sigma_{k+1}^2 C_1^*\hat{C}\hat{P} - \sigma_{k+1}\Sigma B_1\hat{B}^* \\
&\qquad + (\Sigma B_1 + \sigma_{k+1}C_1^*\hat{D})\sigma_{k+1}\hat{B}^* + \sigma_{k+1}^2 C_1^*\hat{W}\hat{L}^*\} \\
&= \sigma_{k+1}^2\Gamma^{-1}C_1^*\{\hat{C}\hat{P} + \hat{D}\hat{B}^* + \hat{W}\hat{L}^*\} \\
&= 0 \quad (\text{by (A.8)}).
\end{aligned}$$

$$\text{partition } (4,1) = -\sigma_{k+1}\hat{B}B_1^* + \sigma_{k+1}\hat{B}B_1^* = 0.$$

$$\text{partition } (4,2) = \sigma_{k+1}\hat{B}B_2^* = 0 \quad (\text{by (A.11)}).$$

$$\begin{aligned}
\text{partition } (4,3) &= \{-\sigma_{k+1}\hat{B}B_1^*\Sigma + \sigma_{k+1}^2\hat{P}\hat{C}^*C_1 + \sigma_{k+1}\hat{B}(B_1^*\Sigma + \sigma_{k+1}\hat{D}^*C_1) \\
&\qquad + \sigma_{k+1}^2\hat{L}\hat{W}^*C_1\}\Gamma^{-1} \\
&= \sigma_{k+1}^2\{\hat{P}\hat{C}^* + \hat{B}\hat{D}^* + \hat{L}\hat{W}^*\}C_1\Gamma^{-1} \\
&= 0 \quad (\text{by (A.8)}).
\end{aligned}$$

$$\begin{aligned}
\text{partition } (4,4) &= \sigma_{k+1}^2(\hat{A}\hat{P} + \hat{P}\hat{A}^* + \hat{B}\hat{B}^* + \hat{L}\hat{L}^*) \\
&= 0 \quad (\text{by (A.8)}).
\end{aligned}$$

The $(2,1)$ and therefore also the $(1,2)$ blocks of (2.33) are verified next.

$$\text{partition } (1,1) = \sigma_{k+1}\hat{D}B_1^* + C_1\Sigma - C_1\Sigma - \sigma_{k+1}\hat{D}B_1^* = 0.$$

$$\text{partition } (1,2) = \sigma_{k+1}(\hat{D}B_2^* + C_2) = 0 \quad (\text{by (A.10)}).$$

$$\begin{aligned}
\text{partition } (1,3) &= \{\sigma_{k+1}\hat{D}(B_1^*\Sigma + \sigma_{k+1}\hat{D}^*C_1) + C_1\Sigma^2 - \sigma_{k+1}^2 C_1 \\
&\qquad - (C_1\Sigma + \sigma_{k+1}\hat{D}B_1^*)\Sigma + \sigma_{k+1}^2\hat{W}\hat{W}^*C_1\}\Gamma^{-1} \\
&= \sigma_{k+1}^2\{\hat{D}\hat{D}^* - I + \hat{W}\hat{W}^*\}C_1 \\
&= 0 \quad (\text{by (A.8)}).
\end{aligned}$$

$$\text{partition } (1,4) = \sigma_{k+1}^2\{\hat{D}\hat{B}^* + \hat{C}\hat{P} + \hat{W}\hat{L}^*\} = 0 \quad (\text{by (A.8)}).$$

The $(2,2)$ block of (2.33) follows immediately from (A.8); (2.33) is thus proven. The validity of (2.34) is established in the same way—in this case use is made of equations (A.1), (A.2), (A.9), (A.16) and (A.18). Since the calculations are very similar to those used to establish (2.33), these details are omitted. □

Appendix B.

Proof of Theorem 3.2. As one would expect, the proofs associated with problems (i) to (iii) are similar. We will therefore only prove the result in the case of (i); the sensitivity proof being marginally more intricate than the others.

Equation (2.6) shows that the $P(s)$ matrix associated with the unweighted sensitivity problem is

$$P(s) = \begin{bmatrix} I & G \\ I & G \end{bmatrix}(s) \tag{B.1}$$

which has a state-space realization

$$P(s) = \left[\begin{array}{c|cc} A & 0 & B \\ \hline C & I & D \\ C & I & D \end{array}\right](s). \tag{B.2}$$

After scaling as in (2.19), we get

$$P(s) = \left[\begin{array}{ccc} A & 0 & BD^{-1} \\ \hline C & I & I \\ C & I & I \end{array}\right](s). \tag{B.3}$$

We have already established that the interpolation points are the right half plane zeros of $G(s)$ and that the left half plane zeros play no part. For simplicity, we will assume that $G(s)$ has all its zeros in $\mathbb{C}_+$. If this is not the case, the Riccati equation balancing theory of this section may be used to reduce the general problem to one in which $\mathrm{Re}[\lambda(A-BD^{-1}C)]>0$. We leave the details to the reader.

If we now substitute the various partitions of (B.3) into (2.20) to (2.23) we obtain

$$X(A-BD^{-1}C)+(A-BD^{-1}C)^*X-XBD^{-1}D^{-*}B^*X=0, \tag{B.4}$$

$$YA^*+AY-YC^*CY=0, \tag{B.5}$$

$$F=C+D^{-*}B^*X, \tag{B.6}$$

$$H=YC^*. \tag{B.7}$$

Since $\mathrm{Re}[\lambda(A-BD^{-1}C)]>0$ by assumption, the stabilizing solution X to (B.4) is nonsingular.

Next, we use (B.3) and (B.4) to (B.7) in (2.28) to obtain

$$T_{12}^*T_{11}T_{21}^*=\left[\begin{array}{cc|c} -\{A-BD^{-1}(C+D^{-*}B^*X)\}^* & XBD^{-1}CY & XBD^{-1} \\ 0 & -\{A-YC^*C\}^* & C^* \\ \hline D^{-*}B^* & -D^{-*}B^*XY & I \end{array}\right]. \tag{B.8}$$

Introducing the basis change

$$T=\begin{bmatrix} I & -XY \\ 0 & I \end{bmatrix} \tag{B.9}$$

gives

(B.10)

$$T_{12}^*T_{11}T_{21}^*=\left[\begin{array}{cc|c} -\{A-BD^{-1}(C+D^{-*}B^*X)\}^* & 0 & X(BD^{-1}-YC^*) \\ 0 & -\{A-YC^*C\}^* & C^* \\ \hline D^{-*}B^* & 0 & I \end{array}\right]$$

$$=\left[\begin{array}{c|c} -\{A-BD^{-1}(C+D^{-*}B^*X)\}^* & X(BD^{-1}-YC^*) \\ \hline D^{-*}B^* & I \end{array}\right].$$

The fact that the $(1,2)$ block of the A-matrix in (B.10) is zero may be proved using (B.4) Y and X(B.5).

The equation defining the observability gramian of $T_{12}^*T_{11}T_{21}^*(s)$ is

$$\begin{aligned} &-\{A-BD^{-1}(C+D^{-*}B^*X)\}Q-Q\{A-BD^{-1}(C+D^{-*}B^*X)\}^* \\ &\quad +BD^{-1}D^{-*}B=0. \end{aligned} \tag{B.11}$$

Comparison of (B.11) with $X^{-1}(\mathrm{B}.4)X^{-1}=0$ reveals that

$$Q=-X^{-1}<0. \tag{B.12}$$

The equation defining the controllability gramian of $T_{12}^*T_{11}T_{21}^*(s)$ is

$$\begin{aligned} &\{A-BD^{-1}(C+D^{-*}B^*X)\}^*P+P\{A-BD^{-1}(C+D^{-*}B^*X)\} \\ &\quad -X(BD^{-1}-YC^*)(D^{-*}B^*-CY)X=0. \end{aligned} \tag{B.13}$$

Substituting PX^{-1}(B.4) and (B.4)$X^{-1}P$ into (B.13) we get

$$\begin{aligned} &X(A-BD^{-1}C)X^{-1}P+PX^{-1}(A-BD^{-1}C)^*X \\ &\quad +X(BD^{-1}-YC^*)(D^{-*}B^*-CY)X=0. \end{aligned} \tag{B.14}$$

Substitution of (B.4) and (B.5) into (B.14) now shows that (B.13) is satisfied by

$$P = -X(I+YX) < 0, \tag{B.15}$$

whence

$$PQ = I + XY$$

and

$$\|(T_{12}^* T_{11} T_{21}^*(s)\|_H = \lambda_{\max}(PQ)^{1/2} = [1+\lambda_{\max}(YX)]^{1/2}.$$

Finally, we know (from the discussion of § 2) that

$$\inf_{K \in \Xi} \|(I+GK)^{-1}(s)\|_\infty = \|(T_{12}^* T_{11} T_{21}^*)^*(s)\|_H,$$

and this concludes the proof of (i). Parts (ii) and (iii) and the unweighted version of (iv) can be proved using similar calculations. To prove the weighted version of (iv), one may invoke the ideas in [13] whereby a weighted optimal robustness problem can be transformed into an equivalent unweighted problem. These details are left to the interested reader. □

Appendix C.

Proof of Theorem 4.1. We may assume without loss of generality that the realizations for $A_1(s)$ and $A_2(s)$ are balanced. Since they are minimal also, the following six all-pass equations [12] are satisfied.

$$A_{11} + A_{11}^* + B_{12}B_{12}^* = 0, \tag{C.1}$$

$$A_{11} + A_{11}^* + C_{11}^* C_{11} = 0, \tag{C.2}$$

$$C_{11} + B_{12}^* = 0, \tag{C.3}$$

$$A_{22} + A_{22}^* + B_{21}B_{21}^* = 0, \tag{C.4}$$

$$A_{22} + A_{22}^* + C_{22}^* C_{22} = 0, \tag{C.5}$$

$$C_{22} + B_{21}^* = 0. \tag{C.6}$$

The first part of the proof will be concerned with the extraction of a maximal degree all-pass left factor $A_l(s)$ from $A_1(s)$ and $G(s)$. Let us consider

$$A_1^* G(s) = \left[\begin{array}{c|c} -A_{11}^* & C_{11}^* \\ \hline -B_{12}^* & I \end{array}\right] \left[\begin{array}{cc|c} A_{11} & A_{12} & B_{11} \\ 0 & A_{22} & B_{21} \\ \hline C_{11} & C_{12} & D \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -A_{11}^* & C_{11}^* C_{11} & C_{11}^* C_{12} & C_{11}^* D \\ 0 & A_{11} & A_{12} & B_{11} \\ 0 & 0 & A_{22} & B_{21} \\ \hline -B_{12}^* & C_{11} & C_{12} & D \end{array}\right]. \tag{C.7}$$

Introducing the change of basis

$$T = \begin{bmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and then making use of (C.2) and (C.3) we obtain

$$A_1^* G(s) = \left[\begin{array}{cc|c} -A_{11}^* & A_{12} + C_{11}^* C_{12} & C_{11}^* D + B_{11} \\ 0 & A_{22} & B_{21} \\ \hline -B_{12}^* & C_{12} & D \end{array}\right]. \tag{C.8}$$

Our purpose now is to show that all the uncontrollable modes in $A_1^*G(s)$ are the poles of $A_1^*(s)$. Since (A_{22}, B_{21}) is controllable by assumption, every uncontrollable mode in (C.8) is an eigenvalue of $-A_{11}^*$. First, we observe that without loss of generality, we may assume that the state-space basis of the realization in (4.15) has been chosen so that in addition to (C.1)-(C.6), we have

(C.9) $$A_{12}+C_{11}^*C_{12}=0.$$

To show that this is so, consider the following change of basis in (4.15)

(C.10) $$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} I & T \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & -T \\ 0 & I \end{bmatrix},$$

(C.11) $$[\bar{C}_{11} \quad \bar{C}_{12}] = [C_{11} \quad C_{12}] \begin{bmatrix} I & -T \\ 0 & I \end{bmatrix}.$$

We now demonstrate that T may be chosen to make

(C.12) $$\bar{A}_{12}+\bar{C}_{11}^*\bar{C}_{12}=0.$$

From (C.10) and (C.11), (C.12) is equivalent to

(C.13) $$(A_{12}+TA_{22}-A_{11}T)+C_{11}^*(C_{12}-C_{11}T)=0.$$

Using (C.2), (C.13) becomes

(C.14) $$A_{11}^*T+TA_{22}+(A_{12}+C_{11}^*C_{12})=0.$$

Since A_{11} and A_{22} are stable, (C.14) always admits a unique solution in T. Such a T ensures that the transformed state-space matrices satisfy (C.12). For notational simplicity, we assume that this basis change has been carried out initially and we revert to the original notation. In view of (C.9), (C.8) becomes

(C.15) $$A_1^*G(s) = \left[\begin{array}{cc|c} -A_{11}^* & 0 & C_{11}^*D+B_{11} \\ 0 & A_{22} & B_{21} \\ \hline -B_{12}^* & C_{12} & D \end{array}\right].$$

If the realization (C.15) has any uncontrollable modes (which must be eigenvalues of $-A_{11}^*$) we may introduce a basis change

(C.16) $$T = \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix}$$

in which U_1 is orthogonal, to transform (C.15) to [24]

$$A_1^*G(s) = \left[\begin{array}{ccc|c} -\tilde{A}_{00}^* & 0 & 0 & \tilde{C}_{10}^*D+\tilde{B}_{01} \\ -\tilde{A}_{01}^* & -\tilde{A}_{11}^* & 0 & \tilde{C}_{11}^*D+\bar{B}_{11} \\ 0 & 0 & A_{22} & B_{21} \\ \hline -\tilde{B}_{02}^* & -\tilde{B}_{12}^* & C_{12} & D \end{array}\right],$$

in which all the uncontrollable modes are eigenvalues of $-\tilde{A}_{00}^*$. That is

(C.17) $$\tilde{C}_{10}^*D+\tilde{B}_{01}=0$$

and thus

(C.18) $$A_1^*G(s) = \left[\begin{array}{cc|c} -\tilde{A}_{11}^* & 0 & \tilde{C}_{11}^*D+\bar{B}_{11} \\ 0 & A_{22} & B_{21} \\ \hline -\tilde{B}_{12}^* & C_{12} & D \end{array}\right]$$

is a controllable realization.

Introducing the basis of (C.18) into (4.15) allows us to write

$$[G \quad A_1](s)=\left[\begin{array}{ccc|cc} \tilde{A}_{00} & \tilde{A}_{01} & \tilde{A}_{02} & \tilde{B}_{01} & \tilde{B}_{02} \\ 0 & \tilde{A}_{11} & \bar{A}_{12} & \bar{B}_{11} & \tilde{B}_{12} \\ 0 & 0 & A_{22} & B_{21} & 0 \\ \hline \tilde{C}_{10} & \tilde{C}_{11} & C_{12} & D & I \end{array}\right] \tag{C.19}$$

in which we have by (C.9)

$$\begin{bmatrix} \tilde{A}_{02} \\ \bar{A}_{12} \end{bmatrix}+\begin{bmatrix} \tilde{C}_{10}^* \\ \tilde{C}_{11}^* \end{bmatrix} C_{12}=0. \tag{C.20}$$

Noting that orthogonal transformations map balanced realizations into balanced realizations in the case of minimal realizations of inner matrices gives

$$\begin{bmatrix} \tilde{A}_{00} & \tilde{A}_{01} \\ 0 & \tilde{A}_{11} \end{bmatrix}+\begin{bmatrix} \tilde{A}_{00}^* & 0 \\ \tilde{A}_{01}^* & \tilde{A}_{11}^* \end{bmatrix}+\begin{bmatrix} \tilde{C}_{10}^* \\ \tilde{C}_{11}^* \end{bmatrix}[\tilde{C}_{10}|\tilde{C}_{11}]=0, \tag{C.21}$$

$$[\tilde{C}_{10}|\tilde{C}_{11}]+[\tilde{B}_{02}^*|\tilde{B}_{12}^*]=0. \tag{C.22}$$

Substituting for $\tilde{A}_{01}$ from (C.21), $\tilde{A}_{02}$ from (C.20), $\tilde{B}_{01}$ from (C.17) and $\tilde{B}_{02}^*$ from (C.22) into (C.19) we get

$$[G \quad A_1](s)=\left[\begin{array}{ccc|cc} \tilde{A}_{00} & -\tilde{C}_{10}^*\tilde{C}_{11} & -\tilde{C}_{10}^*C_{12} & -\tilde{C}_{10}^*D & -\tilde{C}_{10}^* \\ 0 & \tilde{A}_{11} & \bar{A}_{12} & \bar{B}_{11} & \tilde{B}_{12} \\ 0 & 0 & A_{22} & B_{21} & 0 \\ \hline \tilde{C}_{10} & \tilde{C}_{11} & C_{12} & D & I \end{array}\right] \tag{C.23}$$

$$=\left[\begin{array}{c|c} \tilde{A}_{00} & -\tilde{C}_{10}^* \\ \hline \tilde{C}_{10} & I \end{array}\right]\left[\begin{array}{cc|cc} \tilde{A}_{11} & \bar{A}_{12} & \bar{B}_{11} & \tilde{B}_{12} \\ 0 & A_{22} & B_{21} & 0 \\ \hline \tilde{C}_{11} & C_{12} & D & I \end{array}\right] \tag{C.24}$$

$$=A_l(s)[\bar{G} \quad \tilde{A}_1](s) \tag{C.25}$$

where $A_l(s)$ and $\tilde{A}_1(s)$ are as given in (4.17) (note that $\tilde{B}_{02}=-\tilde{C}_{10}^*$). It follows immediately from (C.21) and (C.22) that $A_l(s)$ and hence also $\tilde{A}_1(s)$ are inner.

By using dual arguments, we can extract a maximal degree all-pass right factor $A_r(s)$ from $\bar{G}(s)$ and $A_2(s)$. We begin this calculation with the change of basis

$$T=\begin{bmatrix} I & S \\ 0 & I \end{bmatrix} \tag{C.26}$$

in the state-space of

$$\begin{bmatrix} \bar{G} \\ A_2 \end{bmatrix}(s)=\left[\begin{array}{cc|c} \tilde{A}_{11} & \bar{A}_{12} & \bar{B}_{11} \\ 0 & A_{22} & B_{21} \\ \hline \tilde{C}_{11} & C_{12} & D \\ 0 & C_{22} & I \end{array}\right] \tag{C.27}$$

where S is the unique solution of

$$\tilde{A}_{11}S+SA_{22}^*-\bar{A}_{12}-\bar{B}_{11}B_{21}^*=0. \tag{C.28}$$

The purpose of this basis change is to transform the realization (C.27) to

$$\begin{bmatrix} \bar{G} \\ A_2 \end{bmatrix}(s)=\left[\begin{array}{cc|c} \tilde{A}_{11} & \hat{A}_{12} & \tilde{B}_{11} \\ 0 & A_{22} & B_{21} \\ \hline \tilde{C}_{11} & \hat{C}_{12} & D \\ 0 & C_{22} & I \end{array}\right] \tag{C.29}$$

in which

$$\hat{A}_{12}+\tilde{B}_{11}B_{21}^{*}=0. \tag{C.30}$$

Next, a second orthogonal transformation

$$T=\begin{bmatrix} I & 0 \\ 0 & U_2 \end{bmatrix}$$

together with arguments which are duals of those invoked previously (see equations (C.15) to (C.25)) allow us to write

$$\begin{bmatrix} \bar{G} \\ A_2 \end{bmatrix}(s)=\left[\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_{11} \\ 0 & \tilde{A}_{22} & \tilde{B}_{21} \\ \hline \tilde{C}_{11} & \tilde{C}_{12} & D \\ 0 & \tilde{C}_{22} & I \end{array}\right]\left[\begin{array}{c|c} \tilde{A}_{33} & \tilde{B}_{31} \\ \hline -\tilde{B}_{31}^{*} & I \end{array}\right] \tag{C.31}$$

$$=\begin{bmatrix} \tilde{G} \\ \tilde{A}_2 \end{bmatrix}(s)A_r(s) \tag{C.32}$$

in which both $\tilde{A}_2(s)$ and $A_r(s)$ are inner. Further,

$$\bar{G}A_2^{*}(s)=\tilde{G}\tilde{A}_2^{*}(s)$$

$$=\left[\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12}+\tilde{B}_{11}\tilde{B}_{21}^{*} & \tilde{B}_{11} \\ 0 & -\tilde{A}_{22}^{*} & -\tilde{C}_{22}^{*} \\ \hline \tilde{C}_{11} & \tilde{C}_{12}+D\tilde{B}_{21}^{*} & D \end{array}\right] \tag{C.33}$$

is an observable realization.

Equations (4.17), (4.18), (4.19) and parts (a) and (b) of the theorem have now been established and it remains for us to prove (c). Multiplying (C.33) on the left by $\tilde{A}_1^{*}(s)$ and using (C.21) and (C.22) we get

$$\tilde{A}_1^{*}\tilde{G}\tilde{A}_2^{*}(s)=\left[\begin{array}{cc|c} -\tilde{A}_{11}^{*} & \tilde{A}_{12}+\tilde{B}_{11}\tilde{B}_{21}^{*}+\tilde{C}_{11}^{*}(\tilde{C}_{12}+D\tilde{B}_{21}^{*}) & \tilde{C}_{11}^{*}D+\tilde{B}_{11} \\ 0 & -\tilde{A}_{22}^{*} & -\tilde{C}_{22}^{*} \\ \hline -\tilde{B}_{12}^{*} & \tilde{C}_{12}+D\tilde{B}_{21}^{*} & D \end{array}\right]. \tag{C.34}$$

The minimality of the realization in (C.34) is established by first showing that it is observable. Using (C.21) and (C.22) we have that

$$\begin{aligned}&\left[\begin{array}{cc} sI+\tilde{A}_{11}^{*} & -\tilde{A}_{12}-\tilde{B}_{11}\tilde{B}_{21}^{*}-\tilde{C}_{11}^{*}(\tilde{C}_{12}+D\tilde{B}_{21}^{*}) \\ 0 & sI+\tilde{A}_{22}^{*} \\ \hline -\tilde{B}_{12}^{*} & \tilde{C}_{12}+D\tilde{B}_{21}^{*} \end{array}\right] \\ &=\begin{bmatrix} I & 0 & -\tilde{C}_{11}^{*} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\left[\begin{array}{cc} sI-\tilde{A}_{11} & -A_{12}-\tilde{B}_{11}\tilde{B}_{21}^{*} \\ 0 & sI+\tilde{A}_{22}^{*} \\ \hline \tilde{C}_{11} & \tilde{C}_{12}+D\tilde{B}_{21}^{*} \end{array}\right].\end{aligned} \tag{C.35}$$

Since the $[A, C]$ pair in (C.33) is observable, and because the polynomial matrices on both sides of (C.35) have the same Smith form, the realization in (C.34) is also observable [19].

The minimality of the realization in (C.34) can now be established by showing that it is controllable. We note that

$$\text{(C.36)}\quad \left[\begin{array}{cc|c} sI+\tilde{A}_{11}^* & -\tilde{A}_{12}-\tilde{B}_{11}\tilde{B}_{21}^*-\tilde{C}_{11}^*(\tilde{C}_{12}+D\tilde{B}_{21}^*) & \tilde{C}_{11}^*D+\tilde{B}_{11} \\ 0 & sI+\tilde{A}_{22}^* & -\tilde{C}_{22}^* \end{array}\right] = \left[\begin{array}{cc|c} sI+\tilde{A}_{11}^* & -\tilde{A}_{12}-\tilde{C}_{11}^*\tilde{C}_{12} & \tilde{C}_{11}^*D+\tilde{B}_{11} \\ 0 & sI-\tilde{A}_{22} & \tilde{B}_{21} \end{array}\right]\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\tilde{B}_{21}^* & I \end{bmatrix}.$$

The first matrix on the right of (C.36) is the $[sI-A, B]$ pair of a controllable realization of $\tilde{A}_1^*\tilde{G}(s)$. To show this we assemble from (C.34), (C.31) and (C.32)

$$\text{(C.37)}\quad [\tilde{G}\quad \tilde{A}_1(s)] = \left[\begin{array}{cc|cc} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_{11} & \tilde{B}_{12} \\ 0 & \tilde{A}_{22} & \tilde{B}_{21} & 0 \\ \hline \tilde{C}_{11} & \tilde{C}_{12} & D & I \end{array}\right]$$

and thus

$$\text{(C.38)}\quad \tilde{A}_1^*\tilde{G}(s) = \left[\begin{array}{cc|c} -\tilde{A}_{11}^* & \tilde{A}_{12}+\tilde{C}_{11}^*\tilde{C}_{12} & \tilde{C}_{11}^*D+\tilde{B}_{11} \\ 0 & \tilde{A}_{22} & \tilde{B}_{21} \\ \hline -\tilde{B}_{12}^* & \tilde{C}_{12} & D \end{array}\right].$$

Hence

$$\text{(C.39)}\quad \begin{aligned} \tilde{A}_1^*\tilde{G}A_r(s) &= \tilde{A}_1^*(s)\bar{G}(s) \\ &= \left[\begin{array}{cc|c} -\tilde{A}_{11}^* & \tilde{A}_{12}+\tilde{C}_{11}^*\tilde{C}_{12} & \tilde{C}_{11}^*D+\tilde{B}_{11} \\ 0 & \tilde{A}_{22} & \tilde{B}_{21} \\ \hline -\tilde{B}_{12}^* & \tilde{C}_{12} & D \end{array}\right]\left[\begin{array}{cc} \tilde{A}_{33} & \tilde{B}_{31} \\ -\tilde{B}_{31}^* & I \end{array}\right]. \end{aligned}$$

This cascade realization is system similar (in the sense defined in [19]) to the controllable realization in (C.18) and thus both realizations in (C.39) are controllable. This shows that the realization of (C.38) is controllable as required. The controllability and thus minimality of (C.34) now follows. This completes the proof of the minimality of (4.23). □

Appendix D.

Proof of Theorem 4.2. Since the realizations for $A_1(s)$ and $A_2(s)$ in (4.30) are assumed to be minimal and balanced, equations (C.1) to (C.6) are again satisfied. The proof of part (a) follows by a direct calculation which is similar to the analysis contained in the proof of Theorem 4.1 and is consequently omitted.

To prove part (b), we will need a number of equations which can be deduced from various partitions of (4.33) to (4.36). Since $\tilde{P}$ and $\tilde{Q}$ in (4.33) and (4.34) are symmetric, we may introduce the notation

$$\text{(D.1)}\quad \tilde{P} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^* & P_{22} & P_{23} \\ P_{13}^* & P_{23}^* & P_{33} \end{bmatrix}, \qquad \tilde{Q} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^* & Q_{22} & Q_{23} \\ Q_{13}^* & Q_{23}^* & Q_{33} \end{bmatrix}$$

where the partitioning is conformable with $\tilde{A}$ of (4.32) in which $\underline{A}$ is partitioned as in (4.31). The $(2,2)$ partition of the $(1,1)$ block of (4.33) gives

$$\text{(D.2)}\quad -A_{22}^*P_{22}-P_{22}A_{22}+C_{22}^*C_{22}=0$$

and this together with (C.5) $\Rightarrow P_{22}=-I$. The $(3,2)$ partition of the $(1,1)$ block of (4.33) gives

$$\text{(D.3)}\quad \hat{A}P_{23}^*-P_{23}^*A_{22}+\hat{B}C_{22}=0.$$

The (1, 1) partition of the (1, 1) block of (4.34) gives

$$-A_{11}Q_{11}-Q_{11}A_{11}^*+B_{12}B_{12}^*=0 \tag{D.4}$$

and this together with (C.1) $\Rightarrow Q_{11}=-I$. The (1, 3) partition of the (1, 1) block of (4.34) gives

$$-A_{11}Q_{13}+Q_{13}\hat{A}+B_{12}\hat{C}=0. \tag{D.5}$$

Making use of (C.3), the (1, 1) partition of the (1, 2) block of (4.34) gives

$$B_{12}\hat{D}-B_{11}+Q_{12}C_{22}^*+Q_{13}\hat{B}=0. \tag{D.6}$$

The (2, 1) partition of (4.35) gives

$$Q_{13}P_{23}^*=P_{12}+Q_{12}. \tag{D.7}$$

If we make use of (C.6), the (1, 2) partition of the (2, 1) block of (4.33) yields

$$-\hat{D}C_{22}-B_{12}^*P_{12}+C_{12}-\hat{C}P_{23}^*=0. \tag{D.8}$$

Finally, the (1, 2) partition of the (1, 1) block of (4.34) together with (C.3) gives

$$-A_{11}Q_{12}+A_{12}+B_{11}B_{21}^*-Q_{12}A_{22}^*=0. \tag{D.9}$$

By direct calculation we obtain

$$(G-A_1QA_2)(s)=\left[\begin{array}{cc|c} A_{11} & A_{12} & B_{11} \\ 0 & A_{22} & B_{21} \\ \hline C_{11} & C_{12} & D \end{array}\right]-\left[\begin{array}{ccc|c} A_{11} & B_{12}\hat{D}C_{22} & B_{12}\hat{C} & B_{12}\hat{D} \\ 0 & A_{22} & 0 & B_{21} \\ 0 & \hat{B}C_{22} & \hat{A} & \hat{B} \\ \hline C_{11} & \hat{D}C_{22} & \hat{C} & \hat{D} \end{array}\right]. \tag{D.10}$$

The rest of the proof is based on detailed manipulations of the state-space realizations of $A_1QA_2(s)$ in (D.10). First, we introduce the change of basis

$$T=\begin{bmatrix} I & 0 & Q_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{D.11}$$

and this together with (D.5) yields

$$A_1QA_2(s)=\left[\begin{array}{ccc|c} A_{11} & (B_{12}\hat{D}+Q_{13}\hat{B})C_{22} & 0 & B_{12}\hat{D}+Q_{13}\hat{B} \\ 0 & A_{22} & 0 & B_{21} \\ 0 & \hat{B}C_{22} & \hat{A} & \hat{B} \\ \hline C_{11} & \hat{D}C_{22} & \hat{C}-C_{11}Q_{13} & \hat{D} \end{array}\right]. \tag{D.12}$$

Next, the coordinate transformation

$$T=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -P_{23}^* & I \end{bmatrix} \tag{D.13}$$

together with (D.3) gives

$$A_1QA_2(s)=\left[\begin{array}{ccc|c} A_{11} & (B_{12}\hat{D}+Q_{13}\hat{B})C_{22} & 0 & B_{12}\hat{D}+Q_{13}\hat{B} \\ 0 & A_{22} & 0 & B_{21} \\ 0 & 0 & \hat{A} & \hat{B}-P_{23}^*B_{21} \\ \hline C_{11} & \hat{D}C_{22}+\hat{C}P_{23}^*-C_{11}Q_{13}P_{23}^* & \hat{C}-C_{11}Q_{13} & \hat{D} \end{array}\right]. \tag{D.14}$$

A third change of basis

(D.15)
$$T=\begin{bmatrix} I & -Q_{12} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

leads to

(D.16)

$$A_1QA_2(s)$$
$$=\left[\begin{array}{ccc|c} A_{11} & (B_{12}\hat{D}+Q_{13}\hat{B})C_{22}-Q_{12}A_{22}+A_{11}Q_{12} & 0 & B_{12}\hat{D}+Q_{13}\hat{B}-Q_{12}B_{21} \\ 0 & A_{22} & 0 & B_{21} \\ 0 & 0 & \hat{A} & \hat{B}-P_{23}^*B_{21} \\ \hline C_{11} & \hat{D}C_{22}+\hat{C}P_{23}^*-C_{11}Q_{13}P_{23}^*+C_{11}Q_{12} & \hat{C}-C_{11}Q_{13} & \hat{D} \end{array}\right].$$

Substituting (D.6)C_{22} and (D.9) into the $(1,2)$ block of the A-matrix in the above realization together with (C.5) and (C.6) gives

(D.17)
$$(B_{12}\hat{D}+Q_{13}\hat{B})C_{22}-Q_{12}A_{22}+A_{11}Q_{12}=A_{12}.$$

Substituting (D.6) and (C.6) into the $(1,1)$ block of the B-matrix in (D.16), we obtain

(D.18)
$$B_{12}\hat{D}+Q_{13}\hat{B}-Q_{12}B_{21}=B_{11}.$$

Finally, (D.7), (D.8) and (C.3) will verify that

(D.19)
$$\hat{D}C_{22}+\hat{C}P_{23}^*-C_{11}Q_{13}P_{23}^*+C_{11}Q_{12}=C_{12}.$$

Thus

(D.20)
$$A_1QA_2(s)=\left[\begin{array}{ccc|c} A_{11} & A_{12} & 0 & B_{11} \\ 0 & A_{22} & 0 & B_{21} \\ 0 & 0 & \hat{A} & \hat{B}-P_{23}^*B_{21} \\ \hline C_{11} & C_{12} & \hat{C}-C_{11}Q_{13} & \hat{D} \end{array}\right].$$

Consequently

(D.21)
$$(G-A_1QA_2)(s)=\left[\begin{array}{c|c} \hat{A} & \hat{B}-P_{23}^*B_{21} \\ \hline -\hat{C}-C_{11}Q_{13} & D-\hat{D} \end{array}\right]$$

and this proves (b)(i). Part (b)(ii) is obvious. □

Appendix E.

Proof of Theorem 4.3. The equations describing the closed loop of Fig. 1 are

$$\dot{x}=Ax+B_1u_1+B_2u_2,$$
$$y_1=C_1x+D_{11}u_1+D_{12}u_2,$$
$$y_2=C_2x+D_{21}u_1+D_{22}u_2,$$
$$\dot{\hat{x}}=\hat{A}\hat{x}+\hat{B}y_2,$$
$$u_2=-(\hat{C}\hat{x}+\hat{D}y_2).$$

Eliminating the variables u_2 and y_2 leads to the following state-space model for the closed loop

$$\text{(E.1)}\quad \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A - B_2\hat{D}MC_2 & -B_2[I - \hat{D}MD_{22}]\hat{C} \\ \hat{B}MC_2 & \hat{A} - \hat{B}MD_{22}\hat{C} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_1 - B_2\hat{D}MD_{21} \\ \hat{B}MD_{22} \end{bmatrix} [u_1],$$

$$\text{(E.2)}\quad [y_1] = [C_1 - D_{12}\hat{D}MC_2 \quad -D_{12}[I - \hat{D}MD_{22}]\hat{C}] \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + [D_{11} - D_{12}\hat{D}MD_{21}][u_1]$$

in which

$$\text{(E.3)}\qquad M := (I + D_{22}\hat{D})^{-1}.$$

If s_0 is an unobservable mode of the closed loop state-space model (E.1)-(E.3), then there exists a vector $[w_1^* w_2^*]^* \neq 0$ such that

$$\text{(E.4)}\qquad \left[\begin{array}{cc} s_0I - A + B_2\hat{D}MC_2 & B_2[I - \hat{D}MD_{22}]\hat{C} \\ -\hat{B}MC_2 & s_0I - \hat{A} + \hat{B}MD_{22}\hat{C} \\ \hline C_1 - D_{12}\hat{D}MC_2 & -D_{12}[I - \hat{D}MD_{22}]\hat{C} \end{array}\right] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0.$$

Defining

$$z_2 := -\hat{D}MC_2w_1 - [I - \hat{D}MD_{22}]\hat{C}w_2$$

we have from (E.4) that

$$\text{(E.5)}\qquad \left[\begin{array}{c|c} s_0I - A & -B_2 \\ \hline C_1 & D_{12} \end{array}\right] \begin{bmatrix} w_1 \\ z_2 \end{bmatrix} = 0.$$

The proof of the (a) part is concluded by establishing that $[w_1^* w_2^*] \neq 0 \Rightarrow [w_1^* z_2^*] \neq 0$. Suppose for contradiction that $[w_1^* z_2^*] = 0$. This implies that

$$\text{(E.6)}\qquad \begin{aligned} &[I - \hat{D}MD_{22}]\hat{C}w_2 = 0 \\ &\Leftrightarrow (I + \hat{D}D_{12})^{-1}\hat{C}w_2 = 0 \\ &\Leftrightarrow \hat{C}w_2 = 0. \end{aligned}$$

We also have from (E.4) that

$$\text{(E.7)}\qquad (s_0I - \hat{A})w_2 = 0.$$

Equations (E.6) and (E.7) taken together contradict the assumed minimality of the realization in (4.47) which proves the (a) condition. The (b) part may be established by a parallel sequence of arguments. □

Appendix F.

Proof of Theorem 4.5. We begin by pointing out that

$$\text{(F.1)}\qquad (4.66) \Rightarrow I(H_{22}; s_0) = -\frac{1}{\pi}\int_{-\infty}^{\infty} [\ln|\det D(j\omega)| \operatorname{Re}(s_0)/\{|j\omega - s_0|^2\}]\, d\omega.$$

Next, by invoking the system of J-unitary equations (4.63) to (4.65), one may verify that (note that if $G(s)$ is J-unitary, so also is $G^*(s)$)

$$\text{(F.2)}\qquad I - E(s)E^*(s) = (D + CU)^{-*}\{I - U^*U\}(D + CU)^{-1}(s)$$

so that

$$\text{(F.3)}\qquad \begin{aligned} \det(I - E^*E)(s) &= \{\det(D + CU)\}^{-2}\det(I - U^*U) \\ &= \{\det(D(I + D^{-1}CU))\}^{-2}\det(I - U^*U) \\ &= \{\det(D)\}^{-2}\{\det(I + H_{22}U)\}^{-2}\det(I - U^*U). \end{aligned}$$

Therefore

$$I(E;\, s_0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \{-2\ln|\det(D)| - 2\ln|\det(I+H_{22}U)|$$

$$+\ln|\det(I+U^*U)|\}\,\mathrm{Re}\,(s_0)/\{|j\omega - s_0|^2\}]\, d\omega \tag{F.4}$$

$$= I(H_{22};\, s_0) + I(U;\, s_0)$$

$$+\frac{1}{\pi}\int_{-\infty}^{\infty} [\ln|\det(I+H_{22}U)|\,\mathrm{Re}\,(s_0)/\{|j\omega - s_0|^2\}]\, d\omega. \tag{F.5}$$

Since $s_0 \in \mathbb{C}_-$, $H_{22}(s) \in RH^\infty_-$ and $U(s) \in RH^\infty_-$ we have by Poisson's integral formula

$$\frac{1}{\pi}\int_{-\infty}^{\infty} [\ln|\det(I+H_{22}U)|\,\mathrm{Re}\,(s_0)/\{|j\omega - s_0|^2\}]\, d\omega = \ln|\det[I+H_{22}(s_0)U(s_0)]|. \tag{F.6}$$

Hence

$$I(E;\, s_0) = I(H_{22};\, s_0) + I(U;\, s_0) + \ln|\det[I+H_{22}(s_0)U(s_0)]| \tag{F.7}$$

which proves the first part.

We now need to prove that

$$I(U;\, s_0) + \ln|\det(I+H_{22}(s_0)U(s_0))| \tag{F.8}$$

attains a minimum at $U_0 = -H^*_{22}(s_0)$. This is obvious when $H_{22}(s_0) = 0$ since $I(U;\, s_0) \geqq 0$ and $I(0;\, s_0) = 0$. Let us now suppose that $H_{22}(s_0) \neq 0$ and consider the *constant* linear fractional map:

$$\Theta_0(U(s)) = (\Theta_{11}U(s) + \Theta_{12})(\Theta_{21}U(s) + \Theta_{22})^{-1} \tag{F.9}$$

where the Θ_{ij} are sub-blocks of the J-unitary matrix Θ given by

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} (I_m - X_0^*X_0)^{-1/2} & -(I_m - X_0^*X_0)^{-1/2}X_0^* \\ -(I_n - X_0X_0^*)^{1/2}X_0 & (I_n - X_0X_0^*)^{1/2} \end{bmatrix} \tag{F.10}$$

where

$$X_0 := -H_{22}(s_0) \in \mathbb{C}^{n\times m}.$$

By applying (F.7) to (F.9) we get

$$I(\Theta_0;\, s_0) = I(X_0;\, s_0) + I(U;\, s_0) + \ln|\det(I+H_{22}(s_0)U(s_0))| \tag{F.11}$$

and substituting (F.11) into (F.7) we get

$$I(E;\, s_0) = I(H_{22};\, s_0) + I(\Theta_0;\, s_0) - I(X_0;\, s_0). \tag{F.12}$$

Since $I(\Theta_0(U(s));\, s_0) \geqq 0$ and $I(\Theta_0(X_0^*);\, s_0) = 0$ we have that the minimum value of $I(E;\, s_0)$ is given by

$$I(E;\, s_0) = I(H_{22};\, s_0) - I(X_0;\, s_0)$$

$$= I(H_{22};\, s_0) + \tfrac{1}{2}\ln|\det(I - H_{22}(s_0)H^*_{22}(s_0))| \quad \text{(by (F.1))}$$

which completes the proof. □

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