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<td>Author(s)</td>
<td>Chesi, G</td>
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<tr>
<td>Citation</td>
<td>The 2012 American Control Conference (ACC 2012), Montréal, Canada, 27-29 June 2012. In American Control Conference Proceedings, 2012, p. 3322-3327</td>
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<tr>
<td>Issued Date</td>
<td>2012</td>
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<td>URL</td>
<td><a href="http://hdl.handle.net/10722/153079">http://hdl.handle.net/10722/153079</a></td>
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On the Estimation and Control of the Domain of Attraction through Rational Lyapunov Functions

Graziano Chesi

Abstract—This paper addresses the estimation and control of the domain of attraction (DA) of equilibrium points through rational Lyapunov functions (LFs). Specifically, continuous-time nonlinear systems with polynomial nonlinearities are considered. The estimation problem consists of computing the largest estimate of the DA (LEDA) provided by a given rational LF. The control problem consists of computing a polynomial static output controller of given degree for maximizing such a LEDA. It is shown that lower bounds of the LEDA in the estimation problem, or the maximum achievable LEDA in the control problem, can be obtained by solving either an eigenvalue problem or a generalized eigenvalue problem with smaller dimension. The conservatism of these lower bounds can be reduced by increasing the degree of some multipliers introduced in the construction of the optimization problems. Moreover, a necessary and sufficient condition for establishing tightness of the found lower bounds is provided. Some numerical examples illustrate the use of the proposed results.

I. INTRODUCTION

Studying the domain of attraction (DA) of equilibrium points is a key problem in nonlinear control systems. In fact, the DA is the set of initial conditions for which the state of the system asymptotically converges to the equilibrium point under consideration. Hence, when dealing with nonlinear control systems, it is not sufficient to establish that the desired equilibrium point is locally asymptotically stable, but one has also to make sure that the initial condition lies inside the DA of such an equilibrium.

It is well-known that the DA is a complex set, which typically does not admit an analytic representation. Therefore, looking for inner estimates with simple shape of this set has become a fundamental issue since long time, see e.g. [1], [2] where classic methods such as Zubov equation and La Salle theorem are discussed, and recent works such as [3], [4] based e.g. on the computation of reachable sets and logical composition of Lyapunov functions (LFs). A common way of dealing with estimates of the DA in nonlinear control systems is based on linear matrix inequality (LMI) optimizations and polynomial LFs (possibly composite), see e.g. [5]–[12] and references therein. Clearly, it would be useful to enlarge the class of LFs that can be used with these methods, and doing this it would be also useful to derive efficient ways of obtaining the estimates of the DA and to provide conditions for establishing the optimality of these estimates.

This paper provides a contribution in this direction, addressing the problem of estimation and control of the DA of equilibrium points through LMI-based techniques and rational LFs. Specifically, continuous-time nonlinear systems with polynomial nonlinearities are considered. The estimation problem consists of computing the largest estimate of the DA (LEDA) provided by a given rational LF. The control problem consists of computing a polynomial static output controller of given degree for maximizing such a LEDA. It is shown that lower bounds of the LEDA in the estimation problem, or the maximum achievable LEDA in the control problem, can be obtained by solving either an eigenvalue problem, which is a convex optimization problem with LMIs, or a generalized eigenvalue problem with smaller dimension, which is a quasi-convex optimization problem with LMIs and a special class of bilinear matrix inequalities (BMIs). These optimization problems are obtained by exploiting Stengle’s Positivstellensatz [13] and by introducing a suitable square matrix representation (SMR) [6] of the polynomials. The conservatism of these lower bounds can be reduced by increasing the degree of some multipliers introduced in the construction of the optimization problems. Moreover, a necessary and sufficient condition for establishing tightness of the found lower bounds is provided through the solution of linear algebra operations. Some numerical examples illustrate the use of the proposed results.

The paper is organized as follows. Section II introduces some preliminaries. Section III describes the proposed strategy. Section IV presents some illustrative examples. Lastly, Section V concludes the paper with some final remarks.

II. PRELIMINARIES

In this section we introduce the problem formulation and some preliminaries about positive polynomials.

A. Problem Formulation

The notation adopted throughout the paper is as follows:

- \( \mathbb{N}, \mathbb{R} \): space of natural numbers (including zero) and space of real numbers;
- \( 0_n \): origin of \( \mathbb{R}^n \);
- \( \mathbb{R}_0^n \): \( \mathbb{R} \setminus \{0_n\} \);
- \( I_n \): identity matrix \( n \times n \);
- \( A' \): transpose of matrix \( A \);
- \( A > 0 (A \geq 0) \): symmetric positive definite (semidefinite) matrix \( A \);
- \( a > 0 (a \geq 0) \): entry-wise positive (nonnegative) vector \( a \);
- \( A \otimes B \): Kronecker product of matrices \( A \) and \( B \);
- \( \mathcal{P}_n \): set of polynomials \( p : \mathbb{R}^n \to \mathbb{R} \);
- \( \partial p \): degree of polynomial \( p(x) \);
- s.t.: subject to.
Let us consider the continuous-time polynomial system
\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + G(x(t))u(t) \\
y(t) &= h(x(t)) \\
x(0) &= x_{\text{init}}
\end{align*}
\] (1)
where \(x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n\) is the state, \(x_{\text{init}} \in \mathbb{R}^n\) is the initial condition, \(u(t) = (u_1(t), \ldots, u_n(t))^T \in \mathbb{R}^n\) is the input, and \(y(t) = (y_1(t), \ldots, y_n(t))^T \in \mathbb{R}^n\) is the output. Moreover, \(f \in \mathcal{P}_n, G \in \mathcal{P}_{n \times n}, \) and \(h \in \mathcal{P}_n^m\).

We consider that the origin is the equilibrium point of interest. The domain of attraction (DA) of the origin is the set of initial conditions for which the state asymptotically converges to the origin, and it is indicated by
\[
D = \left\{ x_{\text{init}} \in \mathbb{R}^n : \lim_{t \to +\infty} x(t) = 0 \right\}.
\] (2)
In the sequel the dependence on the time \(t\) will be omitted for ease of notation.

In this paper we consider the estimation and control of the DA of the origin via rational Lyapunov functions (LFs), i.e.
\[
v(x) = \frac{v_{\text{num}}(x)}{v_{\text{den}}(x)}
\] (3)
where \(v_{\text{num}}, v_{\text{den}} \in \mathcal{P}_n\). Throughout the paper we assume that \(v_{\text{num}}(x)\) and \(v_{\text{den}}(x)\) are respectively positive definite and positive, and that \(v(x)\) is radially unbounded, i.e.
\[
\begin{align*}
v_{\text{num}}(x) &> 0 \quad \forall x \in \mathbb{R}^n_0 \quad \text{and} \quad v_{\text{num}}(0) = 0 \\
v_{\text{den}}(x) &> 0 \quad \forall x \in \mathbb{R}^n \\
\lim_{\|x\| \to \infty} v(x) &= \infty.
\end{align*}
\] (4)
To this end, we introduce the generic sublevel set of \(v(x)\) as
\[
\mathcal{V}(c) = \left\{ x \in \mathbb{R}^n : v(x) \leq c \right\}
\] (5)
where \(c \in \mathbb{R}\).

The problems considered in this paper are as follows.

- **Estimation problem**: to obtain the largest estimate of the DA (LEDA) provided by the LF \(v(x)\), i.e. the set \(\mathcal{V}(\gamma)\) where \(\gamma\) is the solution of the optimization problem
\[
\gamma = \sup_{c, k} c \quad \text{s.t.} \quad \mathcal{V}(c) \subseteq D.
\] (6)

- **Control problem**: to design a polynomial static output controller \(u = k(y)\) enlarging the DA of the origin, where \(k \in \mathcal{P}_{n_u}^m\) is a polynomial function to determine that we express as
\[
k(y) = \begin{pmatrix} k_{11} + k_{12}y_1 & k_{13}y_2 & \ldots & k_{1n}y_n \\ k_{21} & k_{22}y_1 & \ldots & k_{2n}y_n \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \ldots & k_{nn} \end{pmatrix}.
\] (7)
In this problem, \(k(y)\) has to satisfy the condition
\[
f(0_n) + G(0_n)k(h(0_n)) = 0_n
\] (8)
in order to maintain an equilibrium point at the origin. Moreover, we consider that the coefficients of \(k(y)\) are possibly constrained within some given bounds, i.e.
\[
k_{ij} \in [k_{ij}^-, k_{ij}^+] \quad \forall i = 1, \ldots, n_u \quad \forall j = 1, 2, \ldots
\] (9)
for some \(k_{ij}^-, k_{ij}^+ \in \mathbb{R}\). The problem amounts to determining an admissible controller \(k(y)\) of chosen degree that maximizes the estimate \(\mathcal{V}(c)\), i.e.
\[
\gamma = \sup_{c, k} c \quad \text{s.t.} \quad \mathcal{V}(c) \subseteq D
\] (10)
where \(d \in \mathbb{N}\) is the chosen degree of \(k(y)\).

B. Positive Polynomials

Before proceeding let us briefly describe how positive polynomials can be studied through linear matrix inequalities (LMIs). Consider \(p \in \mathcal{P}_n\). We can express \(p(x)\) via the square matrix representation (SMR) introduced in [6] as
\[
p(x) = b_{\text{pol}}(x, m)' (P + L(\alpha)) b_{\text{pol}}(x, m)
\] (11)
where
\[
m = \begin{bmatrix} \partial p \\ \frac{2}{2} \end{bmatrix},
\] (12)
\(b_{\text{pol}}(x, m)\) (called power vector) is a vector containing all monomials of degree less than or equal to \(m\) in \(x\), \(P\) is a symmetric matrix, \(L(\alpha)\) is any linear parametrization of the set
\[
\mathcal{L}_{\text{pol}} = \left\{ L = L' : b_{\text{pol}}(x, m)'Lb_{\text{pol}}(x, m) = 0 \right\}
\] (13)
and \(\alpha\) is a free vector. This representation is also known as Gram matrix method. We denote the matrices in (11) as
\[
P = \text{SMR}_{\text{pol}}(p), \quad P + L(\alpha) = \text{CSMR}_{\text{pol}}(p).
\] (14)
A sufficient condition for establishing whether \(p(x)\) is nonnegative consists of expressing \(p(x)\) as a sum of squares of polynomials (SOS), i.e. requiring the existence of polynomials \(p_1, p_2, \ldots \in \mathcal{P}_n\) such that
\[
p(x) = \sum_{i} p_i(x)^2.
\] (15)
By using the expression (11), one can obtain a sufficient and necessary condition for establishing whether \(p(x)\) is SOS through LMIs: \(p(x)\) is SOS if and only if there exists \(\alpha\) such that
\[
P + L(\alpha) \geq 0.
\] (16)
Condition (16) is an LMI feasibility test, which amounts to solving a convex optimization problem [14].

If \(p(x)\) is a locally quadratic polynomial, i.e. a polynomial without monomials of degree zero and one (as it happens for positive/negative semidefinite/definite polynomials), a more compact power vector can be used in (11), specifically a power vector without the constant monomial. We refer to such a power vector as \(b_{\text{pol}}(x, m)\). Moreover, we denote the corresponding matrices in (11) as
\[
P = \text{SMR}_{\text{qua}}(p), \quad P + L(\alpha) = \text{CSMR}_{\text{qua}}(p).
\] (17)
See e.g. [15]–[17] and references therein for details about SOS polynomials. See also the Matlab toolbox SMRSTOFT [18] for solving basic optimization problems over polynomials with SOS programming including investigations of the domain of attraction.
III. PROPOSED RESULTS

First of all, let us express the controller \( k(y) \) as

\[
k(y) = K b_{pol}(y, d)
\]

(18)

where \( K \) is a constant real matrix of suitable size and \( d \) is the degree of \( k(y) \). Let us observe that \( K = 0 \) in the estimation problem, while \( K \) has to satisfy (8)–(9) in the control problem. Hence, we denote the set of admissible matrices \( K \) with

\[
K = \begin{cases} 
0 & \text{if “estimation problem”} \\
\{ K : (8)–(9) \text{ hold}\} & \text{if “control problem”}
\end{cases}
\]

(19)

Since (8)–(9) impose linear equations and inequalities on the entries on \( K \), it follows that \( K \) is either a point or a convex polytope.

Next, let us obtain the closed-loop description of (1) in the presence of the controller \( u = k(y) \) as

\[
\begin{aligned}
\dot{x} &= f(x) + G(x) K b_{pol}(h(x), d) \\
x(0) &= x_{init}
\end{aligned}
\]

(20)

The following result provides a condition for establishing whether a sublevel set is an inner estimate of the DA of the origin (either in the absence or in the presence of a controller) by testing the positivity of some polynomials.

**Theorem 1:** Let \( \nu : \mathbb{R}^n \to \mathbb{R} \) be a rational function satisfying (3)–(4) and let \( c \in \mathbb{R} \) be positive. Suppose that there exists \( q \in \mathbb{P}_n^2 \) and \( K \in \mathcal{K} \) such that

\[
\begin{cases}
p(x) > 0 \\
q(x) > 0
\end{cases} \quad \forall x \in \mathbb{R}^n_0
\]

(21)

where

\[
p(x) = -q(x)' \begin{pmatrix} w(x) \\ c v_{den}(x) - v_{num}(x) \end{pmatrix}
\]

(22)

and

\[
w(x) = (v_{den}(x) \nabla v_{num}(x) - v_{num}(x) \nabla v_{den}(x))' \cdot (f(x) + G(x) K b_{pol}(h(x), d))
\]

(23)

Then, \( \nu(x) \) is a LF for the origin, and \( \mathcal{V}(c) \subseteq \mathcal{D} \).

**Proof.** Suppose that (21) holds, and let \( x \in \mathcal{V}(c) \setminus \{0_n\} \). Then, from the first inequality it follows that

\[
0 < -q_1(x) w(x) - q_2(x) (c v_{den}(x) - v_{num}(x)) \\
\leq -q_1(x) w(x)
\]

since \( q_2(x) > 0 \) from the second inequality in (21), and since \( \nu(x) \leq c \). Moreover, since \( q_1(x) > 0 \) from the second inequality in (21), this implies that

\[
0 > w(x)
\]

\[
\frac{\dot{v}(x)}{v_{den}(x)^2}
\]

Hence, it follows that \( \dot{v}(x) < 0 \), i.e. \( v(x) \) is a Lyapunov for the origin (since it proves local asymptotical stability of this equilibrium point) and \( \mathcal{V}(c) \subseteq \mathcal{D} \). \( \square \)

Theorem 1 provides a condition for establishing whether \( \mathcal{V}(c) \) is included in the DA, either in the case of uncontrolled system or in the case of controlled one. This condition is based on the introduction of the auxiliary polynomial function \( q(x) \), which acts as a multiplier. Let us observe that this condition does not require a priori knowledge of the fact whether \( \nu(x) \) is LF for the origin: indeed, it is easy to see that (21) cannot be satisfied for any positive \( c \) if \( \nu(x) \) is not a LF for the origin.

The next step is to exploit Stengle’s Positivstellensatz [13] in order to check condition (21). Let us observe that it is not sufficient to require that \( p(x) \), \( q_1(x) \) and \( q_2(x) \) are SOS polynomials since this would ensure that the polynomials are nonnegative only. Hence, we require that the polynomials have positive definite SMR matrices built with respect to a suitable power vector. In general, this is a sufficient condition only for positive definiteness of the polynomials, as being SOS is a sufficient condition only for nonnegativity, see e.g. [15], [17] and references therein.

In order to define such SMR matrices, let us observe that

\[
(21) \quad p(x) \quad q_2(x) \quad \text{are locally quadratic polynomials}
\]

since \( w(0_n) = 0 \). Hence, with \( q \in \mathbb{P}_n^2 \) and \( p(x) \) as in (22), let us define

\[
\begin{aligned}
Q_1 &= SMR_{pol}(q_1) \\
Q_2 &= SMR_{qua}(q_2)
\end{aligned}
\]

(24)

**Theorem 2:** Let \( \nu : \mathbb{R}^n \to \mathbb{R} \) be a rational function satisfying (3)–(4), and let \( c \in \mathbb{R} \) be positive. Define the quantities in (24). Suppose that there exist \( Q, \alpha \) and \( K \in \mathcal{K} \) such that the following LMIs hold:

\[
\begin{cases}
P(c, K, Q) + L(\alpha) > 0 \\
Q > 0 \\
trace(Q_1) = 1
\end{cases}
\]

(25)

Then, \( \nu(x) \) is a LF for the origin, and \( \mathcal{V}(c) \subseteq \mathcal{D} \).

**Proof.** Suppose that (25) holds. Let us observe that the first inequality in (25) implies that \( p(x) > 0 \) for all \( x \in \mathbb{R}^n_0 \) since the power vector used to define \( SMR_{qua}(p) \), \( b_{lin}(x, \cdot) \), is nonzero whenever \( x \) is nonzero. Similarly, one has that \( q_1(x) > 0 \) for all \( x \in \mathbb{R}^n \), and \( q_2(x) > 0 \) for all \( x \in \mathbb{R}^n_0 \). This implies that (21) holds, and from Theorem 1 we conclude the proof. \( \square \)

Theorem 2 shows how (21) can be converted into an LMI feasibility test. Let us observe that the constraint \( trace(Q_1) = 1 \) normalizes the variables involved in the test: in fact,

\[
(25) \quad \text{holds for some } Q \text{ and } \alpha \quad \alpha > 0
\]

(26)

\[
(25) \quad \text{holds for } \rho Q \text{ and } \rho \alpha \text{ for all } \rho > 0.
\]

Let us consider the selection of the degrees of \( q_1(x) \) and \( q_2(x) \). A possibility is to choose them in order to maximize the degrees of freedom in (25) for a fixed degree of \( p(x) \). This is equivalent to require that the degrees of \( q_1(x) \) and \( q_2(x) \)
and \( q_2(x)(c - v(x)) \), rounded to the smallest following even integers, are equal. This can be achieved as follows:

- choose an even degree for \( q_1(x) \);
- set the degree of \( q_2(x) \) as
  \[
  \partial q_2 = 2 \left[ \frac{\partial q_1 + \partial w - \partial v_{num}}{2} \right];
  \]
- hence, the degree of \( p(x) \) is given by
  \[
  \partial p = \partial v_{num} + \partial q_2.
  \]

Theorem 2 can be exploited to either estimate or control the LEDA, i.e. to solve problems (6) and (10). Indeed, from Theorem 2 one can define a natural lower bound of \( \gamma \) in the estimation problem or in the control problem as

\[
\hat{\gamma} = \sup_{c,K,Q,\alpha} \frac{c}{P(c,K,Q) + L(\alpha) > 0 \quad Q > 0 \quad K \in K \quad \text{trace}(Q_1) = 1.}
\]

Let us observe that the computation of this lower bound is not straightforward because the first constraint in (26) is a BMI (due to the product of \( c \) and \( K \) with \( Q \)), which may lead to nonconvex optimization problems.

A way to cope with this problem is to fix \( Q_2 \), since in such a case the constraints in (26) are LMIs. This provides the lower bound

\[
\hat{\gamma}_1 = \sup_{c,K,Q,\alpha} \frac{c}{P(c,K,Q) + L(\alpha) > 0 \quad Q_1 > 0 \quad K \in K \quad \text{trace}(Q_1) = 1.}
\]

Problem (27) is an eigenvalue problem, which is a convex optimization problem with LMIs also known as semidefinite program [14].

Another way to address (26) is to fix \( Q_1 \). This provides the lower bound

\[
\hat{\gamma}_2 = \sup_{c,K,Q_2,\alpha} \frac{c}{P(c,K,Q) + L(\alpha) > 0 \quad Q_2 > 0 \quad K \in K.}
\]

Problem (28) still contains a BMI in the first constraint due to the product of \( c \) with \( Q_2 \). However, it is worth observing that the matrix size and number of scalar variables in (28) are typically smaller than those of (27): in fact, \( q_1(x) \) multiplies \( w(x) \) in \( p(x) \), while \( q_2(x) \) multiplies \( cv_{den}(x) - v_{num}(x) \), and the degree of \( w(x) \) is always greater than the degree of \( cv_{den}(x) - v_{num}(x) \).

The solution of (28) can be obtained via a one-parameter sweep on \( c \) where an LMI feasibility test is solved for each fixed value of \( c \). Another possibility is via a quasi-convex optimization problem. Indeed, for \( \mu \in \mathbb{R} \) let us define the polynomials

\[
\begin{align*}
p_1(x) &= -q(x)' \left( \begin{array}{c} w(x) \\ -v_{num}(x) \end{array} \right) \\
p_2(x) &= q_2(x)v(x) \\
v_0(x) &= v_{den}(x) + \mu v_{num}(x)
\end{align*}
\]

and the SMR matrices

\[
\begin{align*}
P_1(K,Q) + L(\alpha) &= CSMR_{qua}(p_1) \\
P_2(Q_2) &= SMR_{qua}(p_2) \\
V &= SMR_{pol}(\bar{v})
\end{align*}
\]

where \( P_2(Q_2) \) is built such that

\[
\tilde{V} > 0 \quad \text{and} \quad Q_2 > 0 \quad \Rightarrow \quad P_2(Q_2) > 0.
\]

**Theorem 3:** Let \( v : \mathbb{R}^n \to \mathbb{R} \) be a rational function satisfying (3)-(4), and let \( c, \mu \in \mathbb{R} \) be positive. Define the quantities in (30), and assume that \( \tilde{V} > 0 \). Then,

\[
\hat{\gamma}_2 = \frac{z^*}{1 + \mu z^*},
\]

where \( z^* \) is the solution of

\[
\begin{align*}
z &= \inf_{K,Q_2,\alpha,z} z \\
&\text{s.t.} \quad zP_2(Q_2) + P_1(K,Q) + L(\alpha) > 0 \\
&\quad Q_2 > 0 \\
&\quad 1 + \mu z > 0 \\
&\quad K \in K.
\end{align*}
\]

**Proof.** Suppose that the constraints in (33) hold. Let us pre- and post-multiply the first inequality by \( b_{lin}(x, \partial p/2)' \) and \( b_{lin}(x, \partial p/2) \), respectively, where \( x \neq 0 \). We get:

\[
0 < b_{lin}(x, \partial p/2)'(zP_2(Q_2) + P_1(K,Q) + L(\alpha) - b_{lin}(x, \partial p/2)) = zp_2(x) + p_1(x) = zq_2(x)(v_{den}(x) + \mu v_{num}(x)) - q_1(x)'v(x) + q_2(x)v(x) = -q_1(x)w(x) - q_2(x)(-zv_{den}(x) - z\mu v_{num}(x)) = -q_1(x)w(x) - (1 + \mu z)q_2(x) = \left( -\frac{z}{1 + \mu z}v_{den}(x) - v_{num}(x) \right). \]

Hence, the first inequality in (28) coincides with the first inequality in (33) whenever \( q_2(x) \) and \( c \) are replaced by

\[
q_2(x) \rightarrow q_2(x)(1 + \mu z) \\
c \rightarrow -\frac{z}{1 + \mu z}.
\]

Since \( 1 + \mu z \) is positive, it follows that the constraints in (28) are equivalent to those in (33), and hence (32) holds. \( \square \)

Theorem 3 states that the solution of (28) can be found by solving the optimization problem (33) which is a generalized eigenvalue problem which is a quasi-convex optimization.
problem with LMIs and a special class of BMIs. See [14] for details about GEVPs.

Once that the lower bound in (27) or (28) has been found, a natural question concerns their tightness: is this lower bound tight? Let us consider the estimation problem. The following result provides a necessary and sufficient condition for answering to this question.

**Theorem 4:** Suppose that $0 < \gamma < \infty$, and define

$$M = \{x \in \mathbb{R}^n : b_{lin}(x, \partial p/2) \in \ker(M)\}$$  

(34)

where $M$ is the found optimal value of one of the following:

- $P(c, K, Q) + L(\alpha)$ in (27). In this case, set $i = 1$;
- $P(c, K, Q) + L(\alpha)$ in (28). In this case, set $i = 2$;
- $zP_2(Q_2) + P_1(K, Q) + L(\alpha)$ in (33). In this case, set $i = 2$.

Define also

$$M_1 = \{x \in M : v(x) = \hat{\gamma} \text{ and } \hat{\gamma}(x) = 0\}. \quad (35)$$

Then,

$$\hat{\gamma}_i = \gamma \iff M_1 \neq \emptyset. \quad (36)$$

**Proof.** “$\Rightarrow$” Suppose that $\hat{\gamma}_i = \gamma$, and let $x^*$ be the tangent point between the surface $\hat{\gamma}(x) = 0$ and the sublevel set $\mathcal{V}(\gamma)$, i.e.

$$\hat{\gamma}(x^*) = 0 \quad v(x^*) = \gamma.$$

Pre- and post-multiplying $M$ by $b_{lin}(x^*, \partial p/2)'$ and $b_{lin}(x^*, \partial p/2)$, respectively, we get

$$0 \leq b_{lin}(x^*, \partial p/2)' M b_{lin}(x^*, \partial p/2) = -q_1'(x^*)w(x^*) - q_2'(x^*) (\gamma v_{den}(x^*) - v_{num}(x^*)) = 0$$

since $M$ is positive semidefinite for definition of $\hat{\gamma}$, $w(x^*) = 0$ and $\gamma v_{den}(x^*) - v_{num}(x^*) = 0$. This implies that

$$b_{lin}(x^*, \partial p/2) \in \ker(M)$$

i.e. $x^* \in M$. Hence, $x^* \in M_1$, i.e. $M_1 \neq \emptyset$.

“$\Leftarrow$” Suppose that $M_1 \neq \emptyset$ and let $x$ be a point of $M_1$. It follows that $v(x) = \hat{\gamma}_i$ and $\hat{\gamma}(x) = 0$. Since $\hat{\gamma}_i$ is a lower bound of $\gamma$, this implies that $x$ is a tangent point between the surface $\hat{\gamma}(x) = 0$ and the sublevel set $\mathcal{V}(\hat{\gamma}_i)$. Hence, $\hat{\gamma}_i = \gamma$. \hfill $\square$

Theorem 4 provides a condition for establishing whether the lower bounds in (27) and (28) are tight. In particular, this happens if the set $M_1$ in (35) is nonempty. This set can be found via trivial substitution from the set $M$ in (34), which can be computed by solving linear algebra operations as explained e.g. in [17].

Before proceeding it is worth mentioning that a weak point of the strategy proposed in this section is the computational burden, which grows quickly with the dimension of the state, degree of the system and degree of the Lyapunov function. This is a consequence of the fact that we have exploited LMIs to establish whether a polynomial is positive definite, and limits the use of the proposed strategy to small dimension and small degree systems.

### IV. EXAMPLES

In this section we present two illustrative examples of the proposed results. The SMR matrices are built with algorithms similar to those reported in [17].

#### A. Example 1

Let us consider the system described by

$$\begin{cases}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= -3x_1 - 2x_2 + x_1^2
\end{cases}$$

where $x \in \mathbb{R}^2$ is the state. We consider the problem of determining the LEDA of the origin provided by the rational function

$$v(x) = \frac{(2x_1^2 + x_1x_2 + x_2^2)(3x_1^2 + 2x_1x_2 + x_2^2 + 1)}{1 + x_1^2 + x_2^2},$$

Let us compute the lower bound $\hat{\gamma}_1$ in (32). We simply select $\partial q_1 = 0$ in the choice of the degrees reported after Theorem 2, hence finding

$$\hat{\gamma}_2 = 24.179.$$

Next, we investigate the tightness of the found lower bound. To this end, we compute the set $M_1$ in (35). We find

$$M_1 = \{(1.259, 2.289)': \emptyset\}$$

and hence from Theorem 4 we conclude that $\hat{\gamma}_2$ is tight since $M_1$ is nonempty, i.e. $\hat{\gamma}_2 = \gamma$. Figure 1 shows the curve $\hat{\gamma}(x) = 0$, the boundary of the LEDA, and the point in $M_1$.

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**Fig. 1.** Example 1. Curve $\hat{\gamma}(x) = 0$ (dashed), boundary of the LEDA $\mathcal{V}('\text{solid line},$ and point in $M_1$ (’o’ mark).

#### B. Example 2

Let us consider the system described by

$$\begin{cases}
    \dot{x}_1 &= -x_1 + x_2 - x_1^2 - x_2^3 + x_1u \\
    \dot{x}_2 &= -2x_2 - x_1^2 + u \\
    y &= x_1
\end{cases}$$

where $x \in \mathbb{R}^2$ is the state, $u \in \mathbb{R}$ is the input, and $y \in \mathbb{R}$ is the output. We consider the problem of designing a
Theorem 2. We consider first that

select where

the quadratic controller. Lastly, we suppose that

found controllers, and the curve

For all the three found controllers, we test the tightness of

W e repeat the computation supposing that

controller (dashed).

It has been shown that lower bounds of the LEDA in the

nominal systems through LMI-based techniques and rational

control of the DA of equilibrium points of nonlinear poly-

For establishing tightness of the found lower bounds has been

provided.

ACKNOWLEDGEMENT

The author would like to thank the Reviewers for their useful comments.

REFERENCES


