A Macroscopic Approach to the Lane Formation Phenomenon in Pedestrian Counterflow *

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Abstract: This paper simulates the pedestrian counter flow by adopting an optimal path-choice strategy and a recently observed speed-density relationship. Although the whole system is symmetric, the simulation demonstrates the segregation and formation of many walking lanes for two groups of pedestrians. The symmetry breaking is most likely triggered by small numerical viscosity or "noise", and the segregation is associated with the minimization of travel time. The underlying physics can be compared with the "optimal self-organization" mechanism in Helbing's social force model, by which driven entities in an open system tend to minimize their interaction to enable them to reach some ordering state.

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The phenomenon of lane formation in pedestrian flows has been frequently observed through direct observation or controlled experiments. Helbing et al. [4] presented photographs which showed that pedestrians could form uniform walking lanes at sufficiently high densities. Theoretically, the phenomenon was explained through an optimal self-organization mechanism [3, 6], namely, a pedestrian crowd constitutes an open system of driven entities that tends to minimize interaction and dissipation and thus reach an optimal state. This differs from a closed system, which is governed by the second law of thermodynamics and thus the entropy (or disorder) increases continuously.

In the literature, few models are found to reproduce the lane formation phenomenon even though many models (e.g, in [8]) have been proposed to deal with pedestrian counter flow. The social force model [5] is typical among these. Although this model looks completely symmetric with respect to the right and left-hand sides, it could well simulate the phenomenon [3, 4, 6]. This was explained as a symmetry-breaking phenomenon through a noise-induction effect [3, 6]. We comment that, with small "noise", which could be due to small numerical viscosity or errors in the computational scheme, the shear stress has a crucial effect, such that two pedestrians walking in opposite directions will turn aside (uniformly left or right) before making physical contact. Without numerical viscosity, as that in a lattice gas or cellular automata model, the "noise" could also be due to the randomicity in a symmetric lattice gas or cellular automata model, which helps destroy the symmetry of the system

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and thus reproduce the phenomenon [1]. This differs from some introduced turn-aside effect in the model (e.g., see [10, 11]).

In this connection, the present paper adopts a macroscopic approach to reproduce the lane formation phenomenon in pedestrian counter flow. Although the macroscopic approach is much more computationally efficient than the microscopic many-particle and cellular automata models, it is at a disadvantage in providing a detailed description of the complexity of pedestrian behavior and psychology. Nevertheless, the physics for self-organization in our approach can be compared with that in the social force model: it is symmetric, under an optimal principle, and with noise-induction through the small numerical viscosity in the computational scheme, even though there are no apparent links between the two types of formulations.

We consider two groups of pedestrians a and b who have different destinations or exits in a walking facility Ω. The formulation includes three major components: (i) a recently proposed speed-density relationship, which symmetrically takes into account the crossing effect (angle) between the two pedestrian groups [14]; (ii) determination of the directions of motion of the two pedestrian groups at the crossing (x, y, t), which is based on the aforementioned speed-density relationship; and (iii) the mass conservations of the two pedestrian groups.

Let the density $\rho(x, y, t) = \{\rho^a(x, y, t), \rho^b(x, y, t)\}$ (in ped/m^2), and let $\psi(x, y, t) \in [0, \pi]$ be the intersecting angle between the two pedestrian streams at the crossing $(x, y) \in \Omega$ at time t. Then, the speed of the pedestrian stream c is assumed to be [14]

$$v^c(\rho, \psi) = v_f \exp(-\alpha(\rho^a + \rho^b)^2) \exp(-\beta(1 - \cos(\psi))(\rho^c)^2), \tag{1}$$

where v_f is the free-flow speed, α and β are two positive parameters. Obviously, $v^c \to v_f$, for $\rho^a + \rho^b \to 0$, which implies $\rho^c \to 0$; v^c is very close to zero for $\rho^a + \rho^b$ being sufficiently large. If $\psi = 0$, the two streams are actually the same and formula (1) reduces to that in [2]. Because $\cos\psi = \cos(-\psi)$, formula (1) is symmetric such that $\cos\psi$ together with $v^c(\rho, \psi)$ would remain unchanged if the two groups took an opposite direction of motion. For counter flow in a symmetric facility, this means that the resulting system would be symmetric to the original system with respect to the symmetric line of the walking facility, if the origins and destinations of the groups a and b were exchanged. In this case, the new and original systems could be identical through a symmetric transformation.

We also easily see the following properties: (i) $v^a(\rho, \psi) \neq v^b(\rho, \psi)$, for $\psi \neq 0$, unless $\rho^a = \rho^b$; and (ii) $v^a(\rho, \psi) = v^b(\rho, \psi)$, if $\beta = 0$. By property (i), we note that formula (1) becomes symmetric provided that $\rho^a = \rho^b$. By property (ii), however, the formula (with $\beta = 0$) would unreasonably suggest that $v^a = v^b$ even though $\rho^a \neq \rho^b$. According to formula (1), we define the cost distribution of group c:

$$\tau^c(x, y, t) = \tau(v^c(\rho, \psi)) = 1/v^c(\rho, \psi). \tag{2}$$

Suppose that a pedestrian expects a steady flow state, which suggests that ρ and ψ (and thus τ^c) are independent of time t. Then, the cost of a pedestrian in a sufficiently small distance ds is expected to be the travel time $\tau^c(x, y, t)ds = ds/v^c(\rho(x, y, t), \psi(x, y, t))$, and the total travel time along a trajectory l is expected to be

$$\int_l \tau^c(x, y, t)ds = \int_l \tau^c(x, y, t)\sqrt{dx^2 + dy^2} \geq \int_l \tau^c(x, y, t) \cos\theta dx + \tau^c(x, y, t) \sin\theta dy, \tag{3}$$

where $\theta(x, y, t)$ is an arbitrarily given angle, and l starts from the current position P_0 to the destination $P_d^c \in \Gamma_d^c$. Assume that the pedestrian should minimize the expected total travel time. This can be realized if and only if, (i) the integral on the right-hand side in Eq. (3) is constant; and (ii) there exists a path $l = l_0$, by which the equality between the two integrals is achieved. The requirement of (i) implies that the integral is independent of the path l, which is ensured if and only if there exists a function $\phi^c(x,y,t)$, such that $\tau^c\cos\theta = -\phi^c_x$, and $\tau^c\sin\theta = -\phi^c_y$. This leads to the following Eikonal equation:

$$\sqrt{(\phi^c_x(x,y,t))^2 + (\phi^c_y(x,y,t))^2} = \tau^c(x,y,t). \tag{4}$$

The required equality of (ii) is achieved if and only if $dx = ds\cos\theta$, and $dy = ds\sin\theta$. This means that, given that ϕ^c can be properly solved by Eq. (4) for fixed t, the direction $(dx/ds, dy/ds)$ of the path $l = l_0$ is chosen as

$$\mathbf{e}^c(x,y,t) := -\nabla\phi^c(x,y,t)/\tau^c(x,y,t), \tag{5}$$

where $\nabla\phi^c = (\phi^c_x, \phi^c_y)$ is the gradient of the function ϕ^c. In this case, by setting $\phi^c(P^c_d, t) \equiv 0$, the expected minimal total travel time in Eq. (3) turns out to be $\phi^c(P_0,t) \equiv \phi^c(x,y,t)$, which is also called the cost potential in an arbitrarily given location $P_0(x,y)$; the motion direction of a pedestrian at P_0 is also determined by Eq. (5), along which the potential ϕ^c decreases at the fastest speed. Moreover, this helps to determine

$$\cos\psi(x,y,t) = \mathbf{e}^a(x,y,t)\cdot\mathbf{e}^b(x,y,t). \tag{6}$$

See also the discussion in [7, 9], which resulted in the same formulas.

Given the velocity $\mathbf{v}^c(x,y,t) = v^c(x,y,t)\mathbf{e}^c(x,y,t)$, the mass conservations of the two pedestrian streams are easily written as

$$\rho^c_t(x,y,t) + \nabla\cdot\mathbf{F}^c(x,y,t) = 0, \tag{7}$$

where the flow $\mathbf{F}^c(x,y,t) = \rho^c(x,y,t)v^c(\rho(x,y,t),\psi(x,y,t))\mathbf{e}^c(x,y,t)$. We assume the initial-boundary conditions: $\rho^c(x,y,0) = \rho^c_0(x,y)$, for $(x,y)\in\Omega$, $\phi^c(x,y,t) = 0$, for $(x,y)\in\Gamma^c_d$, $\mathbf{F}^c(x,y,t)\cdot\mathbf{n}(x,y) = q^c(x,y,t)$, for $(x,y)\in\Gamma^c_o$, and $\mathbf{F}^c(x,y,t)\cdot\mathbf{n}(x,y) = 0$, for $(x,y)\in\Gamma_w$. Here, Γ^c_o and Γ^c_d are the origin and destination, Γ_w is the wall boundary, and $\mathbf{n}(x,y)$ is the unit vector pointing to Ω and vertical to Γ^c_o or Γ_w.

Equations (1)-(2), and (4)-(7) constitute a complete system. For a grid division (x_i, y_j, t^n) in $\Omega\times[0,T]$, we start from $\{\rho^n_{i,j}\}$ to derive $\{\rho^{n+1}_{i,j}\}$ by the following steps: (i) compute $\tau^c(x_i,y_j,t^n)$ by Eqs.(1)-(2), and (5)-(6); (ii) solve Eq.(4) through the first-order fast sweeping method [16] to derive $\phi^c(x_i,y_j,t^n)$ and thus determine $\mathbf{e}^c(x_i,y_j,t^n)$ by Eq. (5); and (iii) solve Eq.(7) through the third-order WENO (weighted essentially non-oscillation) scheme and the third-order TVD (total variation diminishing) Runge-Kutta time discretization [13] to derive $\rho^{n+1}_{i,j}$. See also [9, 15] for further relevant discussion.

We now consider the pedestrian counter flow on a $100m\times 50m$ platform, which is symmetric to $x = 50m$ and $y = 25m$, and is divided into small grids of the size $0.4m\times 0.4m$. The top and bottom borders are walls, and the left and right borders are open. At $t = 0$, the domain is empty with $\rho^c_0(x,y) = 0$ for $(x,y)\in\Omega$, and thus the direction of motion $\mathbf{e}^c(x,y,0)$ is preliminarily set to point to the destination Γ^c_d. Then, pedestrian groups a and b begin to walk onto the platform from the left and right borders towards the right and left borders, respectively. The flow rate $q^c = tq_c/30$, for $t\le 30$, and $q^c = q_c$, for $t > 30$. We take $v_f = 1.034m/s$ and $\alpha = 0.075$ in all test examples.

By setting $q_a = q_b$ $(= 0.4ped/m/s)$, the system is obviously symmetric to the centerline $x = 50m$ (see also discussions below Eq. (1)), by which one should expect a complete confrontation between the two streams of pedestrians and thus a complete block at the centerline. However, this symmetric solution to our macroscopic model is singular, hence unstable and cannot be obtained from any numerical scheme containing small numerical viscosity. In our simulation, we clearly observe the turn-aside behavior and merging after the encounter (Fig. 1(a)), as if the simulated pedestrians (as particles) were truly smart. The turning aside behavior is not influenced by a specific walking habit in that some pedestrians turn left and others in the same group turn right to avoid collision with pedestrians of the other group. As a consequence, the two pedestrian streams automatically form multiple walking lanes (Figs. 1(b) and (c)). The simulation displays typical symmetry breaking and self-organization phenomena.

There seems to have a grid convergence for the lane patterns when we refine the grid sufficiently. While the numerical discretization would generate fewer and wider lanes with very coarse meshes (not shown here to save space), when the refinement comes to a suitable grid size 0.4m$\times$0.4m which is comparable to that in the cellular automation for pedestrian flow [1], the number of lanes becomes stable and does not change with further mesh refinement (compare Figs. 1(c) and 2(a)). Since the mesh refinement suggests a process for the numerical viscosity to approach to zero, the convergence implies high level sensitivity of the symmetry to break for numerical viscosity. We remark that the symmetry breaking is inherent in the studied system, for which the underlying physics is analogous to that in the social force model [3, 5, 6]), and that similar grid convergence for symmetric breaking solutions is common in computational fluid dynamics (e.g. [12]).

The segregation of lanes is associated with the path-choice strategy (Eqs.(1)-(4)). Since each pedestrian expects an optimal path, the two groups as interacting entities have to coordinate with each other to reach an optimal system, which is referred to as the "optimal self-organization" [3]. As a consequence, the travel time of each pedestrian is minimized and the whole system reaches the expected steady flow state (Figs. 1(c), 2(a) and 2(b)). The numerical experiment also indicates that the second exponential term of Eq.(1) plays a key role in the process of lane formation. The number of lanes (and thus the conflicting interaction between the two groups of pedestrians) increases as β decreases (compare Figs. 1(c) and 2(b)). The formation of lanes takes longer when β is getting smaller, which is not expected to reach when β=0. We conclude that a larger value of β suggests a higher level instability of the symmetric system, or a higher level sensitivity of the symmetry to break for a certain "noise" or numerical viscosity.

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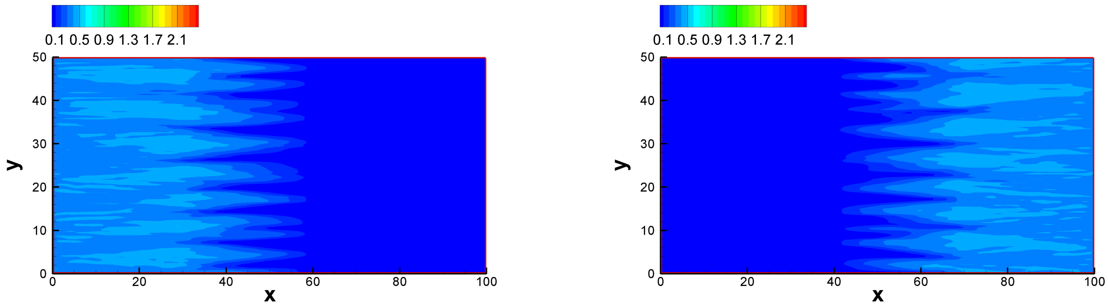
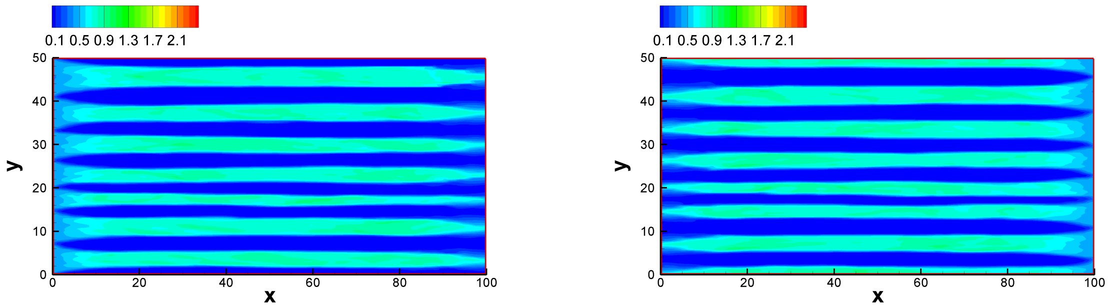
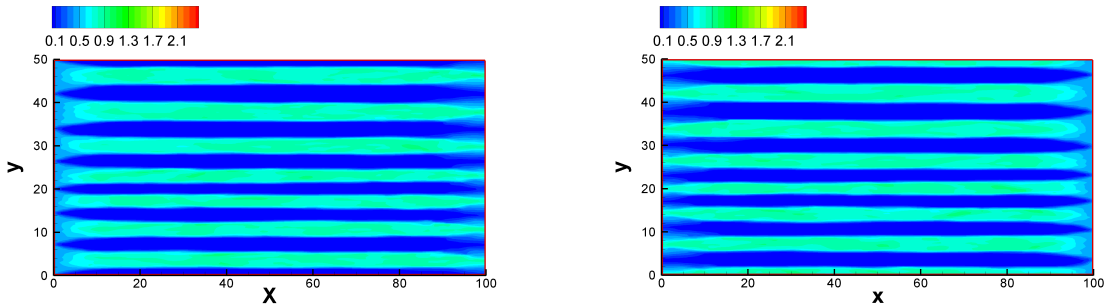


(a) $t = 60$ s

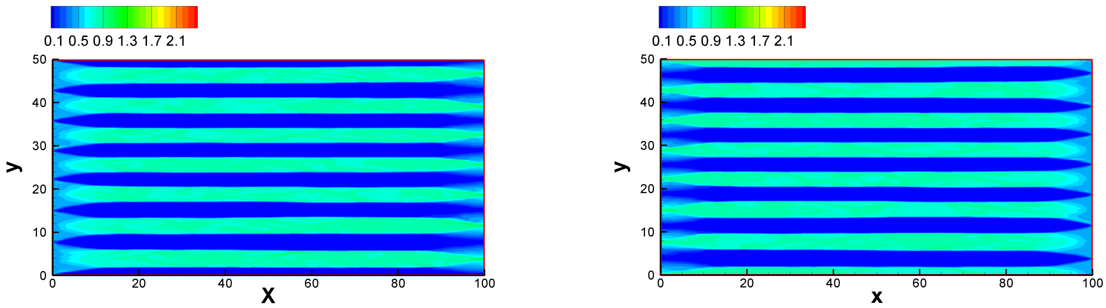


(b) $t = 120$ s

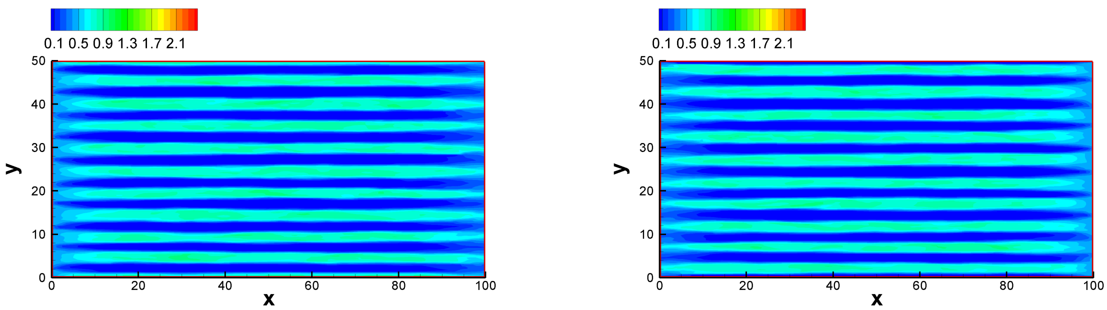


(c) $t = 1800$ s

Figure 1: Density plots for $\beta = 0.019$ and $q_1 = q_2 = 0.4$ with the mesh size: 250×125.



(a) $t = 1800$ s, $\beta = 0.019$, flow level: $(0.4, 0.4)$. Mesh size: 500×250.



(b) $t = 1800$ s, $\beta = 0.009$, flow level: $(0.4, 0.4)$. Mesh size: 250×125.

Figure 2: Density plots for (a) different mesh size; (b) different β, which are compared with Fig. 1.