Optimal reinsurance and dividend for a diffusion model with capital injection: variance premium principle

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Abstract

This paper considers the optimal dividend problem with proportional reinsurance and capital injection for a large insurance portfolio. In particular, the reinsurance premium is assumed to be calculated via the variance principle instead of the expected value principle. Our objective is to maximize the expectation of the discounted dividend payments minus the discounted costs of capital injection. This optimization problem is studied in four cases depending on whether capital injection is allowed and whether there exist restrictions on dividend policies. In all cases, closed-form expressions for the value function and optimal dividend and reinsurance policies are obtained. From the results, we see that the optimal dividend distribution policy is of threshold type with a constant barrier, and that the optimal ceded proportion of risk exponentially decreases with the initial surplus and remains constant when the initial surplus exceeds the dividend barrier. Furthermore, we show that the optimization problem without capital injection is the limiting case of the problem with capital injection when the proportional transaction cost goes to infinity.

Keywords: Capital injection; Dividend optimization; HJB equation; Proportional cost; Proportional reinsurance; Stochastic control; Variance premium principle.

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1 Introduction

In recent years, stochastic control theory has been widely used to tackle optimization problems in the context of insurance risk theory. For details, the readers are referred to Schmidli (2008). In the literature, many papers deal with optimization problems with a combination of risk control and dividend distribution in a diffusion model. For example, Højgaard and Taksar (1999) obtained closed-form solutions to the value function, and the optimal proportional reinsurance and dividend policies; Taksar (2000) presented a survey of stochastic model of risk control and dividend optimization techniques for a financial corporation; Asmussen et al. (2000) studied the excess-of-loss reinsurance policies, and obtained explicit expressions for the value function and optimal stochastic control policies. Subsequently, various generalizations of the problem were studied. Among others, Højgaard and Taksar (2001) considered the problem in the presence of return on investments; Choulli et al. (2003) investigated the problem with constraints on the risk control policies; Taksar and Hunderup (2007) investigated the influence of bankruptcy value on the optimal control policies; and He et al. (2008) considered solvency constraints in terms of ruin probability in the finite-time case. In addition, the optimization problem was also studied in the Cramer-Lundberg model; see, for example, Azcue and Muler (2005) and Mnif and Sulem (2005).

The optimal dividend problem was first studied by de Finetti (1957). Due to its practical importance, much research on dividend-payment problems has been carried out for various surplus processes since then. For recent papers, see Gerber and Shiu (2003, 2004, 2006), Lin and Pavlova (2006), Yuen et al. (2007, 2008, 2009), Dong et al. (2009) and references therein. In addition to dividend payments, Sethi and Taksar (2002) has considered a diffusion model with random returns for a company that can issue new equity when the surplus becomes negative; Kulenko and Schimidli (2008) studied the optimal dividend problem in the Cramer-Lundberg model with capital injection, and obtained closed-form solutions for exponential claims; Løkka and Zervos (2008) derived the optimal dividend and capital injection policies for a diffusion model with proportional costs which show that whether injecting capital or not depends on the size of the proportional cost of capital injection; He and Liang (2008) studied the optimization problem of Løkka and Zervos (2008) with proportional reinsurance under the expected value principle.

When proportional reinsurance is taken as a risk control, the expected value principle is commonly used as the reinsurance premium principle due to its simplicity and popularity in practice. Although the variance principle is another important premium principle, few papers consider using it for risk control in a dynamic setting. Generally speaking, the expected value premium principle is commonly used in life insurance which has the stable and smooth claim frequency and claim sizes, while the variance premium principle is extensively used in property insurance. The variance principle permits the company to take the fluctuations (variance) of claims into consideration when pricing insurance contracts.

Motivated by the work of Løkka and Zervos (2008) and He and Liang

(2008), we study the optimal capital injection and dividend problem with proportional reinsurance under the variance premium principle. The value function is to maximize the expectation of the discounted dividend payments minus the discounted costs of capital injection. Closed-form expressions for the value function and optimal control policies are obtained in four cases depending on whether capital injection is allowed and whether there exist restrictions on dividend policies. Unlike the results obtained under the expected value principle, there exists a common switch level (instead of two levels under the expected value principle; see Højgaard and Taksar (1999)) for the optimal reinsurance and dividend distribution policies. Hence, the optimal dividend policy is of threshold type. In addition, the optimal ceded proportion of risk exponentially decreases with the initial surplus under the variance principle (instead of decreasing linearly under the expected value principle; also see Højgaard and Taksar (1999)), and remains constant when the initial surplus exceeds the switch level.

In this paper, we obtain results different from those in Løkka and Zervos (2008). In the presence of the risk control, it can be shown that the optimization problem without capital injection is the limiting case of the one with capital injection at zero surplus level as the proportional transaction cost of capital injection goes to infinity. Since the value function with capital injection is a decreasing function of the proportional cost, and converges to the value function without capital injection, this suggests that one should allow capital injection regardless of the size of the proportional cost.

The rest of the paper is organized as follows. In Section 2, we introduce the diffusion model with proportional reinsurance and dividend payments under the variance premium principle. Parallel to Højgaard and Taksar (1999), we consider the optimization problem in the case of unrestricted and restricted rates of dividend payments without considering capital injection in Section 3. In Section 4, we carry out a similar study for the optimization problem when capital injection is allowed. Closed-form solutions to the value function and the optimal control policies are obtained in both Sections 3 and 4. Finally, we give some concluding remarks in Section 5.

2 Model formulation

We start with a filtered probability space $(\Omega, \mathcal{F}, \mathcal{G} = \{\mathcal{G}_t, t \geq 0\}, \mathbb{P})$, where the filtration $\mathcal{G}$ satisfies the usual conditions, that is, $\mathcal{G}$ is right continuous and $\mathbb{P}$-completed. Throughout the paper, it is assumed that all stochastic processes and random variables are well-defined on this probability space.

Reinsurance is an important business activity for an insurance company to control its risk position. A reinsurance contract can be represented by a measurable functional $R(\cdot)$ defined on the space composed of all positive random variables such that $0 \leq R(Y) \leq Y$. Under reinsurance R, a positive risk Y is decomposed into two parts, namely $R(Y)$ and $Y - R(Y)$, where $R(Y)$ is retained by the insurer and $Y - R(Y)$ is ceded to the reinsurer. Reinsurance R can be of various forms in theory, but proportional reinsurance and excess-of-loss reinsurance are most commonly used in practice. For proportional reinsurance, $R(Y) = (1-a)Y$, where $0 \leq a \leq 1$ is called the ceded proportion of risk. On the other hand, for excess-of-loss reinsurance, $R(Y) = \min\{Y, b\}$, where $0 \leq b < \infty$ is called the retained level of risk.

Throughout the paper, we assume that the reinsurance premium payments are calculated using the variance principle instead of the expected value principle. For the ceded risk $Y - R(Y)$, the reinsurance premium under the variance principle is given by

$$p(Y - R(Y)) = \mathbb{E}(Y - R(Y)) + \theta\mathbb{D}(Y - R(Y)), \tag{2.1}$$

where $\mathbb{D}$ stands for variance, and $\theta > 0$ is a loading associated with the variance of ceded risk.

In the classical risk theory, the surplus process of an insurance portfolio follows the compound Poisson risk process $\{U(t)\}$ with

$$U(t) = x + ct - \sum_{k=1}^{N(t)} Y_k, \tag{2.2}$$

where x is the initial surplus; c is the rate of premium; $\{N(t)\}$ is a Poisson process with jump intensity λ; and $\{Y_k\}$, independent of $\{N(t)\}$, is a sequence of positive claim-amount random variables with common distribution function F, finite mean μ_1 and finite second moment μ_2^2. Suppose that reinsurance R is taken for each claim. Then, the total ceded risk up to time t is given by $\sum_{k=1}^{N(t)}(Y_k - R(Y_k))$, and the aggregate reinsurance premium under the variance principle takes the form

$$
\begin{aligned}
& \mathbb{E}\left(\sum_{k=1}^{N(t)}(Y_k - R(Y_k))\right) + \theta\mathbb{D}\left(\sum_{k=1}^{N(t)}(Y_k - R(Y_k))\right) \\
= \ & \lambda\left((\mu_1 - \mathbb{E}(R(Y))) + \theta\mathbb{E}((Y - R(Y))^2)\right)t.
\end{aligned}
$$

Similar to the case with the expected value principle, the aggregate reinsurance premium under the variance principle is also proportional to time t. Thus, the surplus process in the presence of reinsurance R can be written as

$$
U^R(t) = x + (c - c^R)t - \sum_{k=1}^{N(t)} R(Y_k), \tag{2.3}
$$

where $c^R = \lambda((\mu_1 - \mathbb{E}(R(Y))) + \theta\mathbb{E}((Y - R(Y))^2))$ represents the reinsurance premium rate associated with R. Here, we assume that the reinsurance market is frictionless. This means that the reinsurance premium rate is equal to the premium rate $c = \lambda(\mu_1 + \theta\mu_2^2)$ if the whole risk are ceded to the reinsurer. Also, it is well known that the jump model is very difficult to deal with for optimal control problems. In view of this, we approximate the model (2.3) by a pure diffusion model $\{X_t^R, t \geq 0\}$ with the same drift and volatility. Specifically, X_t^R satisfies the following stochastic differential equation

$$
dX_t^R = \theta\lambda\left(\mu_2^2 - \mathbb{E}((Y_k - R(Y_k))^2)\right)dt + \sqrt{\lambda\mathbb{E}((R(Y_k))^2)}dB_t, \tag{2.4}
$$

with $X_0^R = x$, where $\{B_t, t \geq 0\}$ is a standard Brownian motion.

In the dynamic setting, it is difficult to consider reinsurance policy with general form. As was shown in Pesonen (1984) and Hipp and Taksar (2010), the proportional reinsurance is optimal in the mean-variance model under the variance principle. In particular, for a given reinsurance R, there exists a proportional parameter $0 \leq a \leq 1$ such that

$$\mathbb{E}((R(Y_k))^2) = \mathbb{E}(((1-a)Y_k)^2) \quad \text{and} \quad \mathbb{E}((Y_k - R(Y_k))^2) > \mathbb{E}((aY_k)^2).$$

Plugging this into (2.4), one can show that the surplus process with proportional reinsurance always has a larger drift under the condition of same volatility. This suggests that proportional reinsurance yields a larger surplus process, and hence more future dividends are expected to be paid. Therefore, proportional reinsurance is also optimal for maximizing the expectation of discounted dividend payments. From now on, R is assumed to be a proportional reinsurance policy with $R(y) = (1-a)y$. Then, we rewrite (2.4) as

$$dX_t^a = (1-a^2)\theta\lambda\mu_2^2 dt + (1-a)\sqrt{\lambda}\mu_2 dB_t, \tag{2.5}$$

with $X_0^a = x$. Associated with this stochastic differential equation, we define the operator $\mathcal{A}^a$ by

$$\mathcal{A}^a f(x) = \frac{1}{2}(1-a)^2\lambda\mu_2^2 f''(x) + (1-a^2)\theta\lambda\mu_2^2 f'(x), \tag{2.6}$$

for any twice continuously differentiable function f.

Assume that the parameter a can be adjusted dynamically to control the risk position, and that dividends may be paid to shareholders. Then, the surplus process with proportional reinsurance and dividend payments is governed by

$$dX_t^\pi = (1-a_t^2)\theta\lambda\mu_2^2 dt + (1-a_t)\sqrt{\lambda}\mu_2 dB_t - dL_t, \tag{2.7}$$

with $X_0^\pi = x$, where L_t is the cumulative dividends paid up to time t and $\pi = (a, L)$ is a control policy. Here, we say a policy π is admissible if

- the ceded proportion $a = \{a_t, t \geq 0\}$ is a $\mathcal{G}$-predictable process with $0 \leq a_t \leq 1$ for all $t \geq 0$, and

- the cumulative amount of dividends $L = \{L_t\}$ is a non-decreasing càdlàg (that is, right continuous with left limit) process, and satisfies the conditions that $L_{0-} = 0$ and $L_t \leq X_t^\pi$ for all $t \geq 0$. The latter condition is required to prohibit dividends from being distributed in the case of deficit.

The set of all admissible control policies is denoted by Π. Under the policy $\pi \in \Pi$, the ruin time of the controlled process X^π is defined as

$$\tau^\pi = \inf\{t \geq 0 : X_t^\pi < 0\}. \tag{2.8}$$

Within this framework, we study the following two cases: (i) the classical optimal dividend problem in which the value function is to maximize the expectation of total discounted dividends until the time of ruin; and (ii) the optimal dividend problem with capital injection in which the value function is to maximize the expectation of the discounted dividend payments minus the discounted costs of capital injection. To tackle these optimal control problems, one needs to solve the associated Hamilton-Jacobi-Bellman (HJB) equations. Since the derivations of the HJB equations and the proofs of verification theorems are standard in the theory of stochastic control, they are omitted in the rest of the paper unless we find them necessary.

Before ending the section, we present the following result which will be repeatedly used in the coming sections:

Lemma 2.1. *Let* $a^*(s) = (\xi + a^*(0))e^{-2\theta s} - \xi,\ s \geq 0$ *such that* $0 \leq a^*(0) \leq 1$. *Suppose that* $H(x)$ *satisfies*

$$H'(x) = \exp\left(\int_x^y 2\theta \frac{a^*(s)}{1-a^*(s)} ds\right),$$

with $2\theta\xi H(0) = (1-a^*(0))H'(0)$ *for some* ξ *and* $y > x \geq 0$. *Then, we have*

$$H(x) = \frac{A(y)}{2\theta\xi}\left(\frac{\xi+1}{\xi+a^*(0)}e^{2\theta x} - 1\right)^{\frac{\xi}{\xi+1}}, \tag{2.9}$$

where

$$A(y) = (\xi + a^*(y))^{\frac{\xi}{\xi+1}}(1-a^*(y))^{\frac{1}{\xi+1}}. \tag{2.10}$$

Note that Lemma 2.1 can be proved by straightforward calculations.

3 Optimization up to time of ruin

In this section, we consider the classical optimal dividend problem of maximizing the expected total discounted dividends until the time of ruin. Under a policy $\pi \in \Pi$, the associated performance function is defined as

$$V^\pi(x) = \mathbb{E}\left(\int_0^{\tau^\pi} e^{-\delta t} dL_t \,\middle|\, X_0^\pi = x\right),$$

where $\delta > 0$ denotes the discounted rate. Then, the value function is given by

$$V(x) = \sup_{\pi\in\Pi} V^\pi(x). \tag{3.1}$$

Our aim is to find expressions for the value function $V(x)$ and the optimal control policy $\pi^* = (a^*, L^*)$ such that $V(x) = V^{\pi^*}(x)$.

Proposition 3.1. *The value function V of (3.1) is concave.*

Proof. See Højgaard and Taksar (1999) for the proof in details. $\square$

As usual, we consider the dividend policy in two cases, namely the case without restrictions and the case with a bounded rate $M < \infty$. As for the two cases, we use V_∞ and V_M to denote the corresponding value functions respectively.

3.1 Unrestricted dividends

In this subsection, we derive explicit expressions for the value function and the optimal policies in the case that no restrictions are imposed on the dividend policy.

Theorem 3.1. *Assume that the value function V_∞ is twice continuously differentiable on $(0, \infty)$. Then, V_∞ satisfies the following HJB equation*

$$\max\left\{\sup_{0\le a\le 1}(\mathcal{A}^a - \delta)V_\infty(x), 1 - V'_\infty(x)\right\} = 0, \tag{3.2}$$

with the boundary condition $V_\infty(0) = 0$. Conversely, if there exists a twice continuously differentiable function $f(x)$ which is a concave solution to (3.2) with the boundary condition $f(0) = 0$, then $f(x) = V_\infty(x)$.

Proof. The proof of this theorem is standard. We refer the reader to Højgaard and Taksar (1999). $\square$

According to the above theorem, in order to find explicit expressions for the value function, we first need to construct a solution to (3.2).

Theorem 3.2. *If there are no restrictions on dividend policy, then the value function V_∞ has the form*

$$V_\infty(x) = \begin{cases} x - b_\infty + \frac{1}{2\theta\xi}, & x \geq b_\infty, \\ \frac{1}{2\theta}\xi^{-\frac{1}{\xi+1}}\left(e^{2\theta x} - 1\right)^{\frac{\xi}{\xi+1}}, & 0 \leq x \leq b_\infty. \end{cases} \tag{3.3}$$

That is, the optimal dividend policy is a barrier dividend strategy with constant barrier b_∞, and the optimal ceded proportion of risk has the form

$$a^*(x) = \begin{cases} 0, & x \geq b_\infty, \\ (\xi+1)\, e^{-2\theta x} - \xi & 0 \leq x \leq b_\infty, \end{cases} \tag{3.4}$$

where

$$b_\infty = \frac{1}{2\theta}\ln\left(\frac{\xi+1}{\xi}\right) \qquad \text{and} \qquad \xi = \frac{\delta}{2\lambda(\theta\mu_2)^2}. \tag{3.5}$$

Proof. Define $b_\infty = \inf\{x \geq 0 : V'_\infty(x) \leq 1\}$. Then, by the concavity of the value function, $V'_\infty(x) > 1$ for $x < b_\infty$. It follows from (3.2) that, for $0 < x < b_\infty$,

$$\sup_{0\leq a\leq 1}\left\{\frac{1}{2}(1-a)^2\lambda\mu_2^2 V''_\infty(x) + (1-a^2)\theta\lambda\mu_2^2 V'_\infty(x)\right\} - \delta V_\infty(x) = 0. \tag{3.6}$$

Differentiating with respect to a and setting the derivative equal to zero, we obtain

$$\frac{V''_\infty(x)}{V'_\infty(x)} = -2\theta\frac{a^*(x)}{1-a^*(x)}, \tag{3.7}$$

where $a^*(x)$ is the maximizer of the first term in (3.6). Putting (3.7) back into (3.6) yields

$$(1-a^*(x))V'_\infty(x) = 2\theta\xi V_\infty(x), \tag{3.8}$$

where $\xi = \delta/(2\lambda(\theta\mu_2)^2) > 0$. This together with the boundary condition $V_\infty(0) = 0$ and the fact that $V'_\infty(x) > 1$ for $x < b_\infty$ give

$$a^*(0) = 1, \tag{3.9}$$

which means that ceding all the risk to the reinsurer is optimal when the initial surplus is zero. On the other hand, taking derivative with respect to x on both sides of (3.8) and using (3.7), we obtain the following ordinary differential equation (ODE)

$$a^{*\prime}(x) = -2\theta a^*(x) - 2\theta\xi, \tag{3.10}$$

which implies $a^{*\prime}(x) < 0$, so the ceded proportion of risk decreases as the initial surplus increases. Solving the ODE of (3.10) with the boundary condition (3.9), one can easily get

$$a^*(x) = (\xi + 1)\, e^{-2\theta x} - \xi. \tag{3.11}$$

Note that we have $a^*(x_0) = 0$ when

$$x_0 = \frac{1}{2\theta} \ln\left(1 + \frac{1}{\xi}\right). \tag{3.12}$$

Assume that $x_0 \le b_\infty$. We next derive expressions for $V_\infty(x)$ for $0 \le x \le x_0$, $x_0 < x \le b_\infty$ and $x > b_\infty$. For $0 \le x \le x_0$, we see from (3.7) that

$$V'_\infty(x) \quad = \quad k \exp\left(\int_x^{x_0} 2\theta \frac{a^*(s)}{1 - a^*(s)} ds\right), \tag{3.13}$$

with $V'_\infty(x_0) = k$, and that

$$V''_\infty(x_0) = -2\theta \frac{a^*(x_0)}{1 - a^*(x_0)} V'_\infty(x_0) = 0,$$

because $a^*(x_0) = 0$. Note that the constant k needs to be determined later. Applying Lemma 2.1 to (3.8) and (3.13) gives the following expression for $V_\infty(x)$

$$V_\infty(x) = \frac{k}{2\theta}\xi^{-\frac{1}{\xi+1}}\left(e^{2\theta x} - 1\right)^{\frac{\xi}{\xi+1}}, \quad 0 \le x \le x_0, \tag{3.14}$$

with $V_\infty(x_0) = k/(2\theta\xi^{-1})$ since $a^*(0) = 1$ and $a^*(x_0) = 0$. For $x_0 < x \le b_\infty$, we take $a^*(x) \equiv 0$ which means that there is no ceded risk in this case. Then, recalling the equation (3.6), we have $V_\infty(x)$ satisfying the following ODE

$$\frac{1}{2}\lambda\mu_2^2 V_\infty''(x) + \theta\lambda\mu_2^2 V_\infty'(x) - \delta V_\infty(x) = 0,$$

which has a solution

$$V_\infty(x) = k_1 e^{r_+(x-b_\infty)} + k_2 e^{r_-(x-b_\infty)}, \quad x_0 < x \le b_\infty, \tag{3.15}$$

where r_+ and r_- are the two roots of the equation

$$\frac{1}{2}\lambda\mu_2^2 r^2 + \theta\lambda\mu_2^2 r - \delta = 0,$$

with $r_+ > 0 > -r_+ > r_-$. Again, the two constants k_1 and k_2 need to be determined later. Finally, for $x \ge b_\infty$, $V_\infty'(x) \equiv 1$ due to (3.2), and $V_\infty(b_\infty) = k_1 + k_2$ because of (3.15). Hence, we have

$$V_\infty(x) = x - b_\infty + k_1 + k_2, \quad x > b_\infty. \tag{3.16}$$

To determine the unknown constants k, k_1, k_2 and b_∞, we need the assumption of twice continuously differentiability of $V_\infty(x)$ which leads to the following four equalities

$$V_\infty'(x_0+) = V_\infty'(x_0-), \quad V_\infty''(x_0+) = V_\infty''(x_0-),$$

$$V_\infty'(b_\infty+) = V_\infty'(b_\infty-), \quad V_\infty''(b_\infty+) = V_\infty''(b_\infty-).$$

From these equalities, we obtain

$$\begin{aligned} k_1 r_+ + k_2 r_- &= 1, && (3.17) \\ k_1 r_+^2 + k_2 r_-^2 &= 0, && (3.18) \\ k_1 r_+ e^{r_+(x_0-b_\infty)} + k_2 r_- e^{r_-(x_0-b_\infty)} &= k, && (3.19) \\ k_1 r_+^2 e^{r_+(x_0-b_\infty)} + k_2 r_-^2 e^{r_-(x_0-b_\infty)} &= 0. && (3.20) \end{aligned}$$

It is easy to check that (3.18) and (3.20) give

$$b_\infty = x_0,$$

and that (3.17) and (3.19) yield $k = 1$. In addition, we have

$$k_1 + k_2 = \frac{r_+ + r_-}{r_+ r_-} = \frac{1}{2\theta\xi}, \tag{3.21}$$

by using the Vieta theorem.

So far, we have shown that the function V_∞ has the form (3.3). But we still need to check that (3.3) is a solution to (3.2) with the boundary condition $V_\infty(0) = 0$. From the construction of V_∞ above, we know that $V_\infty'(x) = 1$ for $x \geq b_\infty$ and that

$$\sup_{0\leq a\leq 1} \left\{ \frac{1}{2}(1-a)^2 \lambda \mu_2^2 V_\infty''(x) + (1-a^2)\theta\lambda\mu_2^2 V_\infty'(x) \right\} - \delta V_\infty(x) = 0, \quad 0 < x < b_\infty.$$

In addition, for $x \geq b_\infty$, we have

$$V_\infty(x) = x - b_\infty + \frac{1}{2\theta\xi},$$

which implies that $V'_\infty(x) = 1$, $V''_\infty(x) = 0$, and

$$\begin{aligned}
&\sup_{0\le a\le 1}\left\{\frac{1}{2}(1-a)^2\lambda\mu_2^2 V''_\infty(x) + (1-a^2)\theta\lambda\mu_2^2 V'_\infty(x)\right\} - \delta V_\infty(x)\\
=\ &\sup_{0\le a\le 1}(1-a^2)\theta\lambda\mu_2^2 - \delta V_\infty(x)\\
=\ &\theta\lambda\mu_2^2 - \delta(x-b_\infty) - \frac{\delta}{2\theta\xi}\\
=\ &-\delta(x-b_\infty) \le 0;
\end{aligned}$$

and, for $0 < x < b_\infty$, we have $V'_\infty(x) > 0$, $V'_\infty(b_\infty) = 1$, and

$$V''_\infty(x) = -2\theta\frac{a^*(x)}{1-a^*(x)}V'_\infty(x) < 0,$$

which implies that $V'_\infty(x) > 1$. Hence, the proof is complete. $\square$

3.2 Dividends with a bounded rate

In this subsection, we impose a restriction that the cumulative dividend process L_t is absolutely continuous and has a bounded density. Specifically, the cumulative dividend process is defined as

$$L_t = \int_0^t l_s ds, \quad 0 \le l_t \le M,$$

for all $t \ge 0$, where $0 < M < \infty$ is a constant. In this case, the problem turns out to be a classical stochastic control problem.

Theorem 3.3. *Assume that the value function V_M is twice continuously differentiable on $(0,\infty)$. Then, V_M satisfies the following HJB equation*

$$\sup_{0\le a\le 1, 0\le l\le M}\{(\mathcal{A}^a - \delta)V_M(x) + l(1 - V'_M(x))\} = 0, \tag{3.22}$$

with the boundary condition $V_M(0) = 0$. Conversely, if there exists a twice continuously differentiable function $f(x)$ which is a concave solution to (3.22) with the boundary condition $f(0) = 0$, then $f(x) = V_M(x)$.

Proof. See Højgaard and Taksar (1999) for the proof in details. $\square$

Because of Theorem 3.3, we need to construct a solution to (3.22). Let $0 < a^* < 1$ be the positive root of the equation

$$a^2 + \left(\xi - 1 + \frac{2M\theta\xi}{\delta}\right) a - \xi = 0, \tag{3.23}$$

that is,

$$a^* = \frac{2\xi}{\sqrt{\left(\xi - 1 + \frac{2M\theta\xi}{\delta}\right)^2 + 4\xi} + \xi - 1 + \frac{2M\theta\xi}{\delta}}. \tag{3.24}$$

Also, define $\tilde{r}_M$ as the negative root of the equation

$$\frac{1}{2}(1 - a^*)^2 \lambda \mu_2^2 r^2 + \left((1 - a^{*2})\theta\lambda\mu_2^2 - M\right) r - \delta = 0,$$

that is,

$$\tilde{r}_M = \frac{2\delta}{(1 - a^{*2})\theta\lambda\mu_2^2 - M - \sqrt{\left((1 - a^{*2})\theta\lambda\mu_2^2 - M\right)^2 + 2(1 - a^*)^2\lambda\mu_2^2\delta}}. \tag{3.25}$$

The two notations a^* and $\tilde{r}_M$ are used in the following theorem, which gives explicit expressions for the value function and the optimal policies.

Theorem 3.4. *If the dividend policy has a density bounded by M with $0 < M < \infty$, then the value function V_M has the form*

$$V_M(x) = \begin{cases} \frac{M}{\delta} + \frac{1}{\tilde{r}_M} e^{\tilde{r}_M (x - b_M)}, & x > b_M, \\ \frac{A}{2\theta\xi}\left(e^{2\theta x} - 1\right)^{\frac{\xi}{\xi+1}}, & 0 \le x \le b_M, \end{cases} \tag{3.26}$$

where $A = (\xi + a^*)^{\frac{\xi}{\xi+1}}(1-a^*)^{\frac{1}{\xi+1}}$. *That is, the optimal dividend policy is a threshold dividend strategy with barrier* b_M *with*

$$l^*(x) = \begin{cases} M, & x > b_M, \\ 0, & 0 < x \le b_M. \end{cases} \tag{3.27}$$

Also, the optimal ceded proportion of risk has the form

$$a^*(x) = \begin{cases} a^*, & x > b_M, \\ (\xi+1)e^{-2\theta x} - \xi, & 0 \le x \le b_M, \end{cases} \tag{3.28}$$

where a^* *is defined by (3.24) and*

$$b_M = \frac{1}{2\theta} \ln\left(\frac{\xi+1}{\xi+a^*}\right). \tag{3.29}$$

Proof. Define $b_M = \inf\{x; V_M'(x) \le 1\}$. Then, by the concavity of the value function, we have $V_M'(x) > 1$ for $0 < x < b_M$. Also, the HJB equation (3.22) turns out to be

$$\sup_{0\le a\le 1}\left\{\frac{1}{2}(1-a)^2\lambda\mu_2^2 V_M''(x) + (1-a^2)\theta\lambda\mu_2^2 V_M'(x)\right\} - \delta V_M(x) = 0, \quad 0 < x < b_M.$$

Now we conjecture that there exists $x_0 \le b_M$ such that $a^*(x) < 1$ for $0 \le x < x_0$ and $a^*(x) \equiv 0$ for $x \ge x_0$. To solve the HJB equation (3.22) for $0 < x < x_0$ and $x_0 < x < b_M$, arguments similar to those in the proof of Theorem 3.2 can be used to obtain

$$\begin{aligned} V_M(x) &= (2\theta\xi)^{-1}(1-a^*(x))V_M'(x), & 0 \le x \le x_0, \\ V_M(x) &= \tfrac{k}{2\theta}\xi^{-\frac{1}{\xi+1}}\left(e^{2\theta x}-1\right)^{\frac{\xi}{\xi+1}}, & 0 \le x \le x_0, \\ V_M(x) &= k_1 e^{r_+(x-b_M)} + k_2 e^{r_-(x-b_M)}, & x_0 < x < b_M. \end{aligned}$$

In addition, for $x \geq b_M$, we have the dividend maximizer $l^*(x) \equiv M$, and hence $V_M(x)$ satisfies the following ODE

$$\frac{1}{2}\lambda\mu_2^2 V_M''(x) + (\theta\lambda\mu_2^2 - M)V_M'(x) - \delta V_M(x) + M = 0. \tag{3.30}$$

Note that $V_M(x)$ is bounded from above by M/δ which is a special solution to (3.30). Thus, $V_M(x)$ has the form

$$V_M(x) = \frac{M}{\delta} + k_3 e^{r_M(x-b_M)}, \tag{3.31}$$

where $r_M < 0$ is the negative root of

$$\frac{1}{2}\lambda\mu_2^2 r^2 + (\theta\lambda\mu_2^2 - M)r - \delta = 0.$$

Since $V'(b_M) = 1$, we have $k_3 = r_M^{-1}$. In order to determine the unknown constants k, k_1, k_2 and b_M, we again employ the assumption of twice continuously differentiability of the value function. Since the value function is smooth at x_0 and b_M, it follows that

$$\begin{aligned} k_1 r_+ + k_2 r_- &= 1, &(3.32)\\ k_1 r_+^2 + k_2 r_-^2 &= r_M, &(3.33)\\ k_1 r_+ e^{r_+(x_0-b_M)} + k_2 r_- e^{r_-(x_0-b_M)} &= k, &(3.34)\\ k_1 r_+^2 e^{r_+(x_0-b_M)} + k_2 r_-^2 e^{r_-(x_0-b_M)} &= 0. &(3.35) \end{aligned}$$

From (3.32) and (3.35), we see that $k_1 > 0$ and $k_2 < 0$. This together with (3.35) imply that

$$0 < e^{(r_+-r_-)(x_0-b_M)} = -\frac{k_2 r_-^2}{k_1 r_+^2} < 1.$$

This inequality yields $k_1 r_+^2 + k_2 r_-^2 > 0$ which contradicts (3.33). Hence, there is no solution to the above system, and the conjecture that $x_0 \leq b_M$ does not hold.

We now conjecture that $b_M < x_0$ such that for $x > x_0$, $a^*(x) \equiv 0$ and dividends are paid at the maximum rate M for $x > b_M$. Then, we again obtain

$$V_M(x) = \frac{M}{\delta} + k_3 e^{r_M(x-x_0)}, \quad x \geq x_0.$$

For $b_M < x < x_0$, the value function satisfies

$$\sup_{0\leq a\leq 1}\left\{\frac{1}{2}(1-a)^2\lambda\mu_2^2 V_M''(x) + (1-a^2)\theta\lambda\mu_2^2 V_M'(x)\right\} - \delta V_M(x) + M(1-V_M'(x)) = 0,$$

which results in

$$\frac{V_M''(x)}{V_M'(x)} = -2\theta\frac{a^*(x)}{1-a^*(x)}.$$

If the conjecture that $b_M < x_0$ is correct, due to the smoothness of the value function at x_0, we should have

$$\frac{V_M''(x_0-)}{V_M'(x_0-)} = \frac{V_M''(x_0+)}{V_M'(x_0+)},$$

but

$$\frac{V_M''(x_0-)}{V_M'(x_0-)} = -2\theta\frac{a^*(x_0)}{1-a^*(x_0)} = 0 \neq r_M = \frac{V_M''(x_0+)}{V_M'(x_0+)}.$$

Thus, the conjecture that $b_M < x_0$ also does not hold. To conclude, there does not exist x_0 and the optimal ceded proportion $a^*(x)$ cannot be zero when the dividend policy has a bounded density.

We now focus on the conjecture that $a^*(x) \equiv a^*$ for all $x > b_M$. We want to determine the switch level b_M and find explicit expressions for $V_M(x)$. According to (3.22), the value function $V_M(x)$ satisfies

$$\frac{1}{2}(1-a^*)^2\lambda\mu_2^2 V_M''(x) + \left((1-a^{*2})\theta\lambda\mu_2^2 - M\right) V_M'(x) - \delta V_M(x) + M = 0, \ \ x > b_M.$$

Since $V_M'(b_M) = 1$ and $V_M(x)$ is bounded, the value function becomes

$$V_M(x) = \frac{M}{\delta} + \frac{1}{\tilde{r}_M} e^{\tilde{r}_M(x-b_M)}, \quad x > b_M, \tag{3.36}$$

where $\tilde{r}_M$ is the negative root of the equation

$$\frac{1}{2}(1-a^*)^2\lambda\mu_2^2 r^2 + \left((1-a^{*2})\theta\lambda\mu_2^2 - M\right) r - \delta = 0. \tag{3.37}$$

In addition, to match the condition

$$\frac{V_M''(b_M-)}{V_M'(b_M-)} = \frac{V_M''(b_M+)}{V_M'(b_M+)},$$

we have

$$-2\theta\frac{a^*}{1-a^*} = \tilde{r}_M. \tag{3.38}$$

Then, by putting (3.38) back into (3.37), we see that $0 < a^* < 1$ is the unique positive root of the equation (3.23). Using (3.11), one can show that b_M is given by (3.29). For $0 \le x \le b_M$, we have

$$V_M'(x) = k\exp\left(\int_x^{b_M} 2\theta\frac{a^*(s)}{1-a^*(s)}ds\right).$$

Then, it follows from $V_M'(b_M) = 1$ that $k = 1$. Noting that $(1-a^*(0))V_M'(0) = 2\theta\xi V_M(0)$ and applying Lemma 2.1, we obtain (3.26) as $a^*(0) = 0$. Finally, from $V_M'(b_M) = 1$, (3.23) and (3.38), we have

$$\begin{aligned} V_M(b_M-) &= \frac{1}{2\theta\xi}(1-a^*)V_M'(b_M) \\ &= \frac{M}{\delta} - \frac{1-a^*}{2\theta a^*} = \frac{M}{\delta} + \frac{1}{\tilde{r}_M} = V_M(b_M+), \end{aligned}$$

which implies that $V_M(x)$ is continuous at $x = 0 \le x \le b_M$.

To end the proof, we need to check that (3.26) is the solution to (3.22). For $x > b_M$, $V_M'(x) = e^{\tilde{r}_M(x-b_M)} < 1$ due to $\tilde{r}_M < 0$, and hence,

$$\begin{aligned} &\sup_{0\le a\le 1, 0\le l\le M}\{(\mathcal{A}^a - \delta)V_M(x) + l(1-V_M'(x))\} \\ &= \sup_{0\le a\le 1}\{(\mathcal{A}^a-\delta)V_M(x)\} + M(1-V_M'(x)) \\ &= \left(\sup_{0\le a\le 1} g(a)\lambda\mu_2^2 - \frac{\delta}{\tilde{r}_M} - M\right)e^{\tilde{r}_M(x-x_1)}, \end{aligned}$$

where $g(a) = \frac{1}{2}(1-a)^2\tilde{r}_M + (1-a^2)\theta$. Then, g attains its maximum at $a = \tilde{r}_M/(\tilde{r}_M - 2\theta)$. On the other hand, comparing this with (3.38), we have $a^* = \tilde{r}_M/(\tilde{r}_M - 2\theta)$. Thus,

$$
\begin{aligned}
&\sup_{0\le a\le 1, 0\le l\le M} \{(\mathcal{A}^a - \delta)V_M(x) + l(1 - V_M'(x))\} \\
&= \left(\frac{1}{2}(1-a^*)^2\lambda\mu_2^2\tilde{r}_M^2 + (1-(a^*)^2)\theta\lambda\mu_2^2\tilde{r}_M - M\tilde{r}_M - \delta\right)\frac{1}{\tilde{r}_M}e^{\tilde{r}_M(x-b_M)} \\
&= 0,
\end{aligned}
$$

where the last step follows from (3.37). For $0 \le x \le b_M$, we know that $V_M'(x) > 1$ due to $V_M'(b_M) = 1$ and

$$
V_M''(x) = -2\theta\frac{a^*(x)}{1-a^*(x)}V_M'(x) < 0.
$$

Thus,

$$
\begin{aligned}
&\sup_{0\le a\le 1, 0\le l\le M} \{(\mathcal{A}^a - \delta)V_M(x) + l(1 - V_M'(x))\} \\
&= \sup_{0\le a\le 1}\left\{\frac{1}{2}(1-a)^2\lambda\mu_2^2V_M''(x) + (1-a^2)\theta\lambda\mu_2^2V_M'(x)\right\} - \delta V_M(x) \\
&= 0,
\end{aligned}
$$

where the last step is based on the construction of the value function in the proof of Theorem 3.2. The proof is complete. □

Remark 3.1. *(Limiting case of maximum dividend rate M) As $M \to \infty$, it follows from (3.24) that $a^* \to 0$ and $b_M \to b_\infty$, and hence $A = A(b_M)$ goes to $\xi^{\frac{\xi}{\xi+1}}$ by (2.10). For $0 < x < b_M$, it is obvious that the solution to the value function in Theorem 3.4 tends to that in Theorem 3.2 as M goes to infinity. Furthermore, for $x \ge b_M$, $\tilde{r}_M \to 0$ as $M \to \infty$. Using Taylor expansion and*

(3.37), we get

$$
\begin{aligned}
V_M(x) &= \frac{M}{\delta} + \frac{1}{\tilde{r}_M} e^{\tilde{r}_M (x - b_M)} \\
&= \frac{(M\tilde{r}_M + \delta) + \delta \tilde{r}_M (x - b_M) + o(\tilde{r}_M (x - b_M))}{\delta \tilde{r}_M} \\
&= \frac{\frac{1}{2}(1-a^*)^2 \lambda \mu_2^2 \tilde{r}_M^2 + (1 - a^{*2})\theta\lambda\mu_2^2 \tilde{r}_M + \delta\tilde{r}_M(x - b_M) + o(\tilde{r}_M(x - b_M))}{\delta\tilde{r}_M} \\
&\to x - b_\infty + \frac{\theta\lambda\mu_2^2}{\delta} = x - b_\infty + \frac{1}{2\theta\xi},
\end{aligned}
$$

as M goes to infinity. That is, the optimal control problem without dividend restrictions can be seen as the limiting optimal control problem with a bounded dividend density when the bound goes to infinity.

4 Optimal dividends with capital injection

Based on the results obtained in Section 3, the optimal dividend policy is a barrier strategy. However, as we all know, ruin will occur almost surely in the presence of barrier dividend policy. Therefore, similar to the work of Kulenko and Schmidli (2008), we allow the investor to inject capital with a proportional transaction cost (or called penalty factor) when the surplus ever becomes negative so that ruin can be avoided. In this section, we maximize the expected discounted dividend payments minus the expected discounted costs of capital injection.

Given an admissible $\pi \in \Pi$, the surplus process can be described by

$$
\tilde{X}_t^\pi = X_t^\pi + Z_t^\pi, \tag{4.1}
$$

where X^π is determined by (2.7) and the capital injection process Z^π can be written as

$$Z_t^\pi = \max\{-\inf_{0\le s\le t} X_s^\pi, 0\},$$

according to Skorohod equation (see Karatzas and Shreve (1991)). The performance function in this case is defined as

$$W^\pi(x) = \mathbb{E}\left(\int_0^\infty e^{-\delta t} d(L_t - \phi Z_t^\pi)\middle| X_0^\pi = x\right), \tag{4.2}$$

where $\delta > 0$ is the discount factor and $\phi > 1$ is the proportional transaction cost for capital injection. Correspondingly, the value function is given by

$$W(x) = \sup_{\pi\in\Pi} W^\pi(x), \tag{4.3}$$

and the objective of this section is to find explicit expressions for $W(x)$ and the optimal policy $\pi^* \in \Pi$ such that $W(x) = W^{\pi^*}(x)$.

Proposition 4.1. *The value function W defined in (4.3) is concave.*

Proof. For any $\varepsilon > 0$, according to the definition (4.3), we can choose sub-optimal admissible control $\pi_i = (a^{(i)}, L^{(i)}) \in \Pi$ for the initial surplus x_i such that

$$W^{\pi_i}(x_i) \ge W(x_i) - \varepsilon, \quad i = 1, 2. \tag{4.4}$$

For any $0 < \rho < 1$, define a new control policy $\pi = (a, L)$ for initial surplus $\rho x_1 + (1-\rho)x_2$ such that

$$\begin{aligned} a_t &= \rho a_t^{(1)} + (1-\rho)a_t^{(2)}, \\ L_t &= \rho L_t^{(1)} + (1-\rho)L_t^{(2)}. \end{aligned}$$

Then, according to (2.7) and Jessen's Inequality, we have $X_t^{\pi} \geq \rho X_t^{\pi_1} + (1-\rho)X_t^{\pi_2}$ with $X_0^{\pi_i} = x_i, i = 1, 2$. In addition, according to (4.1), we have

$$\begin{aligned} Z_t^{\pi} &\leq \max\left\{-\inf_{0\leq s\leq t}\{\rho X_s^{\pi_1} + (1-\rho)X_s^{\pi_2}\}, 0\right\} \\ &\leq \max\left\{-\rho \inf_{0\leq s\leq t} X_s^{\pi_1} - (1-\rho) \inf_{0\leq s\leq t} X_s^{\pi_2}, 0\right\} \\ &\leq \rho \max\left\{-\inf_{0\leq s\leq t} X_s^{\pi_1}, 0\right\} + (1-\rho)\max\left\{-\inf_{0\leq s\leq t} X_s^{\pi_2}, 0\right\} \\ &= \rho Z_t^{\pi_1} + (1-\rho)Z_t^{\pi_2}. \end{aligned}$$

Thus,

$$\begin{aligned} W(\rho x_1 + (1-\rho)x_2) &\geq W^{\pi}(\rho x_1 + (1-\rho)x_2) \\ &\geq \rho W^{\pi_1}(x_1) + (1-\rho)W^{\pi_2}(x_2) \\ &\geq \rho W(x_1) + (1-\rho)W(x_2) - \varepsilon. \end{aligned}$$

Hence, the concavity of W follows from the arbitrariness of ε. □

As was discussed in Section 3, we also consider the dividend policies in two cases: dividends without restrictions and the accumulate dividends with a bound density. We use W_∞ and W_M to denote the corresponding value functions respectively.

4.1 Unrestricted dividends

In this subsection, we first consider the case with no restrictions imposed on the dividend policy.

Theorem 4.1. *Assume that the value function W_∞ is twice continuously differentiable on $(0,\infty)$. Then, $W_\infty(x)$ satisfies the following HJB equation*

$$\max\left\{\sup_{0\le a\le 1}(\mathcal{A}^a-\delta)W_\infty(x), 1-W'_\infty(x)\right\}=0, \tag{4.5}$$

with the boundary condition $W'_\infty(0)=\phi$. Conversely, if there exists a twice continuously differentiable function $f(x)$ which is a concave solution to (4.5) with the boundary condition $f'(0)=\phi$, then $f(x)=W_\infty(x)$.

Proof. Since the derivation of the HJB equation is similar to the classical theory in Fleming and Soner (2006), we just explain the boundary condition $W'_\infty(0)=\phi$. In fact, when the surplus attains zero, the investor injects a small amount of capital ϵ, and then we have

$$W_\infty(0)=W_\infty(\varepsilon)-\phi\varepsilon+o(\varepsilon).$$

Dividing ε on both sides of the above equation and then taking $\varepsilon\to 0+$, it follows that $W'_\infty(x)=\phi$.

For any given admissible policy $\pi=(a,L)$, the surplus process $\tilde{X}^\pi$ is governed by (2.7) and (4.1). Define the first hitting time $\tau_n=\inf\{t\ge 0;\tilde{X}^\pi_t> n\}$. Applying the generalized Itô formula, it follows that

$$\begin{aligned}
e^{-\delta(t\wedge\tau_n)}f(\tilde{X}^\pi_{t\wedge\tau_n}) &= f(x)-\int_0^{t\wedge\tau_n}e^{-\delta s}dL_s+\int_0^{t\wedge\tau_n}e^{-\delta s}f'(\tilde{X}^\pi_s)dZ^\pi_s\\
&\quad+\int_0^{t\wedge\tau_n}(\mathcal{A}^a-\delta)e^{-\delta s}f(\tilde{X}^\pi_s)ds+\int_0^{t\wedge\tau_n}e^{-\delta s}(1-f'(\tilde{X}^\pi_s))dL_s\\
&\quad+\sum_{s\le t}e^{-\delta s}[f(\tilde{X}^\pi_s)-f(\tilde{X}^\pi_{s-})-f'(\tilde{X}^\pi_{s-})(\tilde{X}^\pi_s-\tilde{X}^\pi_{s-})]\\
&\quad+\int_0^{t\wedge\tau_n}e^{-\delta s}f'(\tilde{X}^\pi_s)dB_s.
\end{aligned}$$

Note that $\tilde{X}_t^\pi \geq 0$ for all $t \geq 0$, $dZ_t^\pi = 0$ when $\tilde{X}_t^\pi > 0$, $f(x) - f(y) \leq f'(y)(x-y)$ for any x and y due to the concavity of f, and that the last term in the above equation is a martingale since the integrand is bounded by ϕ. Employing the boundary condition $f'(0) = \phi$ and taking expectation on both sides of the above equation, we obtain

$$f(x) \geq \mathbb{E}\left[e^{-\delta(t\wedge\tau_n)} f(\tilde{X}_{t\wedge\tau_n}^\pi) \mid \tilde{X}_0^\pi = x\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{-\delta s} d(L_s - \phi Z_s^\pi) \mid \tilde{X}_0^\pi = x\right]. \tag{4.6}$$

Taking $n \to +\infty$, by Fatou's lemma and the monotone convergence theorem, we have

$$f(x) \geq e^{-\delta t}\mathbb{E}\left[f(\tilde{X}_t^\pi) \mid \tilde{X}_0^\pi = x\right] + \mathbb{E}\left[\int_0^t e^{-\delta s} d(L_s - \phi Z_s^\pi) \mid \tilde{X}_0^\pi = x\right]. \tag{4.7}$$

Due to the concavity of f, it follows that $f(y) \leq K(1+y)$ for some $K > 0$. In addition, note that

$$\tilde{X}_t^\pi \leq |X_t^a| + \sup_{s\leq t} |X_s^a| \leq 2\sup_{s\leq t} |X_s^a|,$$

where X^a is determined by (2.5) with $X_0^a = x$, that is,

$$X_t^a = x + \int_0^t (1-a_s^2)\theta\lambda\mu_2^2 ds + M_t,$$

where $M_t = \int_0^t (1-a_s)\sqrt{\lambda}\mu_2 dB_s$ is a martingale. Therefore, by applying Doob's maximal inequality (See Karatzas and Shreve (1991)) to submartingale $|M|$, it follows that

$$\begin{aligned}
\mathbb{E}\left[f(\tilde{X}_t^\pi) \mid \tilde{X}_0^\pi = x\right] &\leq \mathbb{E}\left[K(1+\tilde{X}_t^\pi) \mid \tilde{X}_0^\pi = x\right] \\
&\leq K\left(1 + 2x + 2\theta\lambda\mu_2^2 t + 2\mathbb{E}\left[\sup_{s\leq t}|M_s| \mid \tilde{X}_0^\pi = x\right]\right) \\
&\leq K\left(1 + 2x + 2\theta\lambda\mu_2^2 t + 4\left[\mathbb{E}\left(|M_t|^2 \mid \tilde{X}_0^\pi = x\right)\right]^{\frac{1}{2}}\right) \\
&\leq K\left(1 + 2x + 2\theta\lambda\mu_2^2 t + 4\mu_2\sqrt{\lambda t}\right).
\end{aligned}$$

Thus, taking $t \to \infty$ in (4.7) and using the monotone convergence theorem again, we have $f(x) \geq W_\infty^\pi(x)$. Hence, it follows by taking supremum that $f(x) \geq W_\infty(x)$.

Particularly, let $b = \inf\{f'(x) \leq 1\}$. If we take an admissible policy $\pi^* = (a^*, L^*)$ such that dividend distribution policy is a barrier policy with barrier b and reinsurance policy satisfies $(\mathcal{A}^{a^*} - \delta)f(x) = 0$ for $0 < x < b$, then the controlled surplus process $\tilde{X}^{\pi^*}$ is continuous such that $0 \leq \tilde{X}_t^{\pi^*} \leq b$ and $dL_t^* = 0$ for $\tilde{X}^{\pi^*} < b$. Thus, the above inequality (4.6) becomes an equality, that is,

$$f(x) = \mathbb{E}\left[e^{-\delta(t\wedge\tau_n^*)} f(\tilde{X}_{t\wedge\tau_n^*}^{\pi^*}) \mid \tilde{X}_0^{\pi^*} = x\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_n^*} e^{-\delta s} d(L_s^* - \phi Z_s^{\pi^*}) \mid \tilde{X}_0^{\pi^*} = x\right],$$

where $\tau_n^* = \inf\{t \geq 0; \tilde{X}_t^{\pi^*} > n\}$. Note that $0 \leq \tilde{X}_t^{\pi^*} \leq b$ for all $t > 0$. Then, taking $n \to \infty$ and $t \to \infty$ and using the dominated and monotone convergence theorems, we obtain $f(x) = W_\infty^{\pi^*}(x) \leq W_\infty(x)$. Therefore, one can conclude that $f(x) = W_\infty(x)$. □

In the following theorem, we construct a solution to (3.22) and give explicit expressions for the value function.

Theorem 4.2. *If the dividend distribution policy is without any restriction, then the value function W_∞ has the form*

$$W_\infty(x) = \begin{cases} x - \tilde{b}_\infty + \frac{1}{2\theta\xi}, & x > \tilde{b}_\infty, \\ \frac{1}{2\theta}\xi^{-\frac{1}{1+\xi}}\left(\frac{\xi+1}{\xi+a^*(0)}e^{2\theta x} - 1\right)^{\frac{\xi}{1+\xi}}, & 0 \leq x \leq \tilde{b}_\infty. \end{cases} \tag{4.8}$$

That is, the optimal dividend policy is a barrier strategy with dividend barrier

$\tilde{b}_\infty$; and the optimal ceded proportion of risk has the form

$$a^*(x) = \begin{cases} 0, & x > \tilde{b}_\infty, \\ (\xi + a^*(0))e^{-2\theta x} - \xi, & 0 \le x \le \tilde{b}_\infty, \end{cases}$$

where $0 < a^*(0) < 1$ is uniquely determined by the equation

$$\phi^{\xi+1}\left(1 + \frac{a}{\xi}\right)^{\xi}(1-a) = 1, \tag{4.9}$$

and

$$\tilde{b}_\infty = \frac{1}{2\theta}\ln\left(1 + \frac{a^*(0)}{\xi}\right). \tag{4.10}$$

Proof. Recall the steps for solving (3.2) in the last section. Parallel to (3.8), it still follows that

$$(1 - a^*(x))W'_\infty(x) = 2\theta\xi W_\infty(x). \tag{4.11}$$

But with the boundary condition $W'_\infty(0) = \phi$, we cannot directly determine $a^*(0)$ from the above equation without knowing $W_\infty(0)$. Thus, according to (3.10), we only have

$$a^*(x) = (\xi + a^*(0))e^{-2\theta x} - \xi, \tag{4.12}$$

where $a^*(0)$ needs to be determined later. In addition, with the same arguments, we also infer that there is only one switch level, say $\tilde{b}_\infty$, such that $a^*(\tilde{b}_\infty) = 0$, $W'_\infty(\tilde{b}_\infty) = 1$, and $W'_\infty(0) = \phi$. So, parallel to (3.13), we have $k = 1$ and

$$\begin{aligned} (\xi + a^*(0))e^{-2\theta\tilde{b}_\infty} &= \xi, \\ \int_0^{\tilde{b}_\infty} 2\theta\frac{a^*(s)}{1 - a^*(s)}ds &= \ln\phi. \end{aligned}$$

Combining these with (4.12) and doing some basic calculations, one can show that $0 < a^*(0) < 1$ is uniquely determined by the equation (4.9). Then, (4.12) is clear and the switch level $\tilde{b}_\infty$ can be solved using $a^*(\tilde{b}_\infty) = 0$. It follows from (4.11) that

$$W_\infty(0) = \frac{\phi}{2\theta\xi}(1 - a^*(0)). \tag{4.13}$$

Then, parallel to (3.13), we have

$$W'_\infty(x) = k \exp\left(\int_x^{\tilde{b}_\infty} 2\theta \frac{a^*(s)}{1 - a^*(s)} ds\right), \quad 0 \le x \le \tilde{b}_\infty,$$

where $k = 1$ by $W'_\infty(\tilde{b}_\infty) = 1$. Noting that $a^*(\tilde{b}_\infty) = 0$ and $2\theta\xi W_\infty(0) = (1 - a^*(0))$ from (4.11), we put $y = \tilde{b}_\infty$ in Lemma 2.1 to obtain

$$W_\infty(x) = \frac{1}{2\theta}\xi^{-\frac{1}{1+\xi}}\left(\frac{\xi+1}{\xi+a^*(0)}e^{2\theta x} - 1\right)^{\frac{\xi}{1+\xi}}, \quad 0 \le x \le \tilde{b}_\infty,$$

such that $W_\infty(\tilde{b}_\infty) =) = (\xi^{-\frac{1}{1+\xi}}\xi^{-\frac{\xi}{1+\xi}})/(2\theta) = (2\theta\xi)^{-1}$.

Finally, using arguments similar to those in the proof of Theorem 3.2, one can show that (4.8) is a solution to (4.5). So, the proof is complete. □

4.2 Dividends with a bounded rate

Like Subsection 3.2, we consider the case with a bounded dividend rate M in this subsection. The value function in this case is denoted by W_M.

Theorem 4.3. *If the value function W_M is twice continuously differentiable on $(0, \infty)$, then W_M satisfies the following HJB equation*

$$\sup_{0\le a\le 1, 0\le l\le M} \{(\mathcal{A}^a - \delta)W_M(x) + l(1 - W'_M(x))\} = 0, \tag{4.14}$$

with the boundary condition $W'_M(0) = \phi$. Conversely, if there exists a twice continuously differentiable function $f(x)$ which is a solution to (4.14) with boundary condition $f'(0) = \phi$, then $f(x) = W_M(x)$.

Proof. The proof is similar to Theorem 4.1 with slight change so we omit it. $\square$

The following theorem gives explicit expressions for the value function by constructing a solution to (4.14).

Theorem 4.4. *If the cumulative dividend policy has a density bounded by M such that $0 < M < \infty$, then the value function W_M defined in (4.3) has the form*

$$W_M(x) = \begin{cases} \frac{M}{\delta} + \frac{1}{\tilde{r}_M} e^{\tilde{r}_M(x - \tilde{b}_M)}, & x > \tilde{b}_M, \\ \frac{A}{2\theta\xi}\left(\frac{\xi+1}{\xi+a^*(0)} e^{2\theta x} - 1\right)^{\frac{\xi}{1+\xi}}, & 0 \le x \le \tilde{b}_M, \end{cases} \tag{4.15}$$

where $A = (\xi + a^*)^{\frac{\xi}{\xi+1}}(1 - a^*)^{\frac{1}{\xi+1}}$. That is, the optimal dividend policy is a threshold dividend strategy with barrier $\tilde{b}_M$ with

$$l^*(x) = \begin{cases} M, & x > \tilde{b}_M, \\ 0, & 0 < x \le \tilde{b}_M. \end{cases} \tag{4.16}$$

Furthermore, the optimal ceded proportion of risk has the form

$$a^*(x) = \begin{cases} a^*, & x > \tilde{b}_M, \\ (\xi + a^*(0))e^{-2\theta x} - \xi, & 0 \le x \le \tilde{b}_M, \end{cases} \tag{4.17}$$

where a^* is defined by (3.24); and $a^* < a^*(0) < 1$ is the unique solution to the equation

$$\phi^{\xi+1}\left(\frac{\xi + a}{\xi + a^*}\right)^{\xi}\left(\frac{1 - a}{1 - a^*}\right) = 1; \tag{4.18}$$

and

$$\tilde{b}_M = \frac{1}{2\theta}\ln\frac{\xi + a^*(0)}{\xi + a^*}. \tag{4.19}$$

Proof. Solving the HJB equation (4.14) is similar to solve (3.22). Based on the analysis in Theorem 3.4, we conjecture that there is one switch level $\tilde{b}_M$ such that $W_M'(\tilde{b}_M) = 1$, $W_M'(0) = \phi$ and $a^*(x) = a^*(\tilde{b}_M)$ for $x \geq \tilde{b}_M$. Then, parallel to (3.32)-(3.34), we know that $a^*(\tilde{b}_M) = a^*$.

Note that for $0 \leq x \leq \tilde{b}_M$, it still follows that

$$W_M'(x) = k \exp\left(\int_x^{\tilde{b}_M} 2\theta \frac{a^*(s)}{1 - a^*(s)} ds\right),$$

where $k = 1$ by $W_M'(\tilde{b}_M) = 1$. In order to determine $a^*(0)$, we put $y = \tilde{b}_M$ in Lemma 2.1 to obtain (4.15). Then, taking derivative on both sides and applying $W_M'(0) = \phi$, the equation (4.18) follows. Thus, $a^*(0)$ can be uniquely determined by the equation (4.18). Once $a^*(0)$ and $a^*(\tilde{b}_M)$ are worked out, $\tilde{b}_M$ can be obtained using (4.19).

Finally, using arguments similar to those in the proof of Theorem 3.4, we can show that (4.15) is a solution to (4.14). Hence, the proof is complete. □

Remark 4.1. *Similar to Remark 3.1, we can also check that*

$$W_\infty(x) = \lim_{M\to\infty} W_M(x), \quad x \geq 0.$$

That is, with capital injection, the optimal problem without dividend restrictions can also be seen as the limiting case of the optimal problem with a bounded dividend density as the bound goes to infinity.

Before ending this section, we present another remark to compare the value functions obtained in Section 3 and Section 4 respectively.

Remark 4.2. ***The gain of capital injection.*** *According to Remark 3.1 and Remark 4.1, we see that the optimal problem without dividend restrictions is the limiting case of the optimal problem with a bounded dividend density, so we only need to compare the results of the optimization problem with bounded dividend density.*

First, we find that the value functions V_M and W_M satisfy the same HJB equation (see Theorems 3.3 and 4.3) but with different boundary conditions. Without capital injection, ruin occurs immediately when the surplus hits zero so the value function satisfies the boundary condition $V_M(0) = 0$. However, with capital injection with proportional transaction cost $\phi > 1$, the value function satisfies the boundary condition $W_M'(0) = \phi$.

In addition, from the expressions for the value function V_M given by (3.26), we have $a^*(0) = 1$ and

$$V_M'(x) \quad \sim \quad C(e^{2\theta x} - 1)^{-\frac{1}{\xi+1}} e^{2\theta x} \to \infty, \qquad \text{as} \qquad x \to 0, \qquad \text{that is,} \qquad V_M'(0) = \infty.$$

On the other hand, it is easy to verify from (4.18) that

$$a^*(0) \uparrow 1 \qquad \text{and} \qquad \tilde{b}_M \uparrow b_M \qquad \text{as} \quad \phi \to \infty,$$

which imply that

$$W_M(x) \downarrow V_M(x), \qquad \text{as} \quad \phi \to \infty.$$

That is, the optimization problem without capital injection is the limiting case of the problem with capital injection as the proportional cost ϕ tends to *infinity. With capital injection, we can achieve a larger value function than the one without capital injection, regardless of the size of the proportional cost of capital injection. Therefore, we can conclude that it is better to have capital injection when ruin occurs, no matter how large the proportional cost is.*

The result obtained here is somehow different from the result obtained in Løkka and Zervos (2008). The main reason is the existence of proportional reinsurance policies. By adjusting the ceded proportion, the investor can change the drift and volatility of the controlled diffusion process. In fact, no matter how large the proportional cost ϕ is, the investor can always choose $a^*(0)$ such that $W_M(0) > 0$ and $W'_M(0) = \phi$. The two conditions guarantee that having capital injection is better than letting ruin occur.

5 Concluding remarks

In this paper, we investigate the optimal proportional reinsurance and dividend problem for a diffusion model under the variance premium principle instead of the expected value premium principle. The controlled diffusion model is established in terms of the diffusion approximation of the stochastic process. The closed-form expressions for the value functions and the optimal control policies are obtained in four cases depending on whether capital injection is allowed and whether there exist restrictions for dividend policies. The results obtained here under the variance principle are different from those under the expected value principle. The optimal ceded proportion of risk exponentially decreases with respect to the initial surplus. In addition, with the existence of proportional reinsurance policies, the value function with capital injection is always larger than the one without capital injection, regardless of the size of proportional cost of capital injection. In order to obtain closed-form solutions for the value functions and the optimal control policies, cheap reinsurance is an important assumption. The same optimization problem under non-cheap reinsurance assumption is another interesting topic.

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