

# Stochastic oscillations of general relativistic discs

Tiberiu Harko<sup>1,2★</sup> and Gabriela Raluca Mocanu<sup>3★</sup>

<sup>1</sup>Center for Theoretical and Computational Physics, The University of Hong Kong, Pok Fu Lam Road, Hong Kong SAR, China

<sup>2</sup>Department of Physics, The University of Hong Kong, Pok Fu Lam Road, Hong Kong SAR, China

<sup>3</sup>Faculty of Physics, Department of Theoretical and Computational Physics, Babes-Bolyai University, Kogalniceanu Street 1, Cluj-Napoca, Romania

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## ABSTRACT

We analyse the general relativistic oscillations of thin accretion discs around compact astrophysical objects interacting with the surrounding medium through non-gravitational forces. The interaction with the external medium (a thermal bath) is modelled via a friction force and a random force, respectively. The general equations describing the stochastically perturbed discs are derived by considering the perturbations of trajectories of the test particles in equatorial orbits, assumed to move along the geodesic lines. By taking into account the presence of a viscous dissipation and of a stochastic force, we show that the dynamics of the stochastically perturbed discs can be formulated in terms of a general relativistic Langevin equation. The stochastic energy transport equation is also obtained. The vertical oscillations of the discs in the Schwarzschild and Kerr geometries are considered in detail, and they are analysed by numerically integrating the corresponding Langevin equations. The vertical displacements, velocities and luminosities of the stochastically perturbed discs are explicitly obtained for both the Schwarzschild and the Kerr cases.

**Key words:** accretion, accretion discs – black hole physics – gravitation – instabilities.

## 1 INTRODUCTION

The simplest physical model describing the random motion of a non-relativistic Brownian particle is given by the Langevin equation, which can be written as (Coffey, Kalmykov & Waldron 2004)

$$\frac{dp}{dt} = -\nu p + K + \xi(t), \quad (1)$$

where  $p$  is the (non-relativistic) momentum, and the three terms on the right-hand side of equation (1) correspond to the friction, to an external force  $K$  and to a stochastic force  $\xi$ , respectively.  $\nu$  is a coefficient called the friction coefficient. Due to the friction term, the particle in Brownian motion loses energy to the medium, but simultaneously gains energy from the random kicks of the thermal bath, modelled by the random force. The random force, which we assume to be a white noise, satisfies the conditions  $\langle \xi(t) \rangle = 0$  and  $\langle \xi(t)\xi(t') \rangle = k\delta(t - t')$ , where  $k$  is a constant. The separation of the force into frictional and random parts is merely a phenomenological simplification – microscopically, the two forces have the same origin (collision with the external medium constituents). The Langevin equation has a large range of applications in physics, astrophysics, engineering, biology, etc. (Coffey et al. 2004).

An exact formulation of the physics of Barnett relaxation, based on a realistic kinetic model of the relaxation mechanism that in-

cludes the alignment of the grain angular momentum in body coordinates by Barnett dissipation, misalignment by thermal fluctuations and coupling of the angular momentum to the gas via gas damping, was introduced in Lazarian & Roberge (1997). The Fokker–Planck equation for the measure of internal alignment was solved by using numerical integration of the equivalent Langevin equation for Brownian rotation. It was also shown that in a steady state, energy is transferred back to the grain material at an equal rate by Barnett dissipation.

A simple physical model to describe the dynamics of a massive point-like object, such as a black hole, near the centre of a dense stellar system, was developed in Chatterjee, Hernquist & Loeb (2002). The total force on the massive body can be separated into two independent parts, one of which is the slowly varying influence of the aggregate stellar system, and the other being the rapidly fluctuating stochastic force due to discrete encounters with individual stars. The motion of a black hole is similar to that of a Brownian particle in a harmonic potential, and its dynamics can be analysed by using an approach based on the Langevin equation. By numerically solving the Langevin equation, one can obtain the average values, time autocorrelation functions, and probability distributions of the black hole's position and velocity. By using this model a lower limit on the mass of the black hole Sgr A\* in the Galactic Centre can be derived.

A Langevin-type treatment of the motion of a charged particle in the intergalactic medium magnetic field, which allows to estimate both the average and the rms time-delay for particles of given

★E-mail: harko@hkuc.hku.hk (TH); gabriela.mocanu@ubbcluj.ro (GM)

energy, was introduced in Vietri, De Marco & Guetta (2003). This model was compared with a scenario where the particles are accelerated at internal shocks. The generalized thermodynamics and the collapse of a system of self-gravitating Langevin particles exhibiting anomalous diffusion in a space of dimension  $D$  were considered in Chavanis & Sire (2004). The equilibrium states correspond to polytropic distributions. The dynamical stability of stellar systems, gaseous stars and two-dimensional vortices was studied using a thermodynamical analogy.

There are many processes in the accretion of material on to an active galactic nucleus (AGN) that occur on relatively short time-scales as compared to the cooling time of the X-ray-emitting gas in and around elliptical galaxies and galaxy clusters. Accretion rates on to AGNs are likely to be extremely variable on short time-scales. Using a Langevin-type equation, Pope (2007) has shown that, for a simple feedback system, this can induce variability in the AGN power output, which is of much larger amplitude, and persists for longer time-scales, than the initial fluctuations. An implication of this result is that rich galaxy clusters are expected to show the largest and longest lived fluctuations. Stochastic variations in the accretion rate also mean that the AGNs inject energy across a wide range of time-scales.

Accretion discs around compact objects have become standard models for a number of astrophysical phenomena like X-ray binaries or AGNs. The discs are usually considered as being composed of massive test particles that move in the gravitational field of the central compact object. Waves and normal-mode oscillations in geometrically-thin and geometrically-thick discs around compact objects have been studied extensively both within Newtonian gravity (see Kato 2001 for a review) and within a relativistic framework (Okazaki, Kato & Fukue 1987; Perez et al. 1997; Semerak & Zacek 2000; Shi & Li 2000; Kato 2001; Silbergleit, Wagoner & Rodriguez 2001; Rodriguez et al. 2002; Rezzolla, Yoshida & Zanotti 2003; Zanotti et al. 2005; Blaes, Arras & Fragile 2006; O'Neill, Reynolds & Miller 2009). The oscillations of the accretion discs may produce the quasi-periodic oscillations (Titarchuk & Osherovich 2000; Tagger & Varniere 2006; Fan et al. 2008; Horak et al. 2009) in black hole low-mass X-ray binaries.

It is the purpose of this paper to propose a fully general relativistic model for the description of the oscillations of the particles in the accretion discs around compact objects in contact with an external heat bath. The oscillations of the disc, assumed to interact with an external medium (a heat bath), are described by a general relativistic Langevin equation, which is derived, for the case of a general axisymmetric gravitational field, by considering the perturbations of the geodesic equation describing the motion of test particles in stable circular orbits. The energy transport in the presence of the stochastic fluctuations of the disc is discussed in detail. As an application of our general formalism we consider in detail the vertical stochastic oscillations of the accretion discs in both the static Schwarzschild and the rotating Kerr geometries. In these cases the equations of motion of the disc can be formulated as Langevin equations, describing the interaction between the disc and an external thermal bath. By numerically integrating the corresponding Langevin equation we obtain the vertical displacements, velocities and luminosities of the stochastically oscillating discs.

This paper is organized as follows. In Section 2, we briefly review the motion of the test particles in axisymmetric gravitational fields. In Section 3, we consider the stochastic perturbations of the geodesic trajectories of test particles moving around a central compact object. The general system of perturbed equations for stochastic discs is obtained in Section 4. The energy transfer in stochastically perturbed

discs is discussed in Section 5. The vertical motions of the discs are considered in Section 6, and it is shown that they can be described by a Langevin-type equation. The disc oscillation frequencies are also obtained for both the Schwarzschild and the Kerr metrics. The vertical oscillations of the stochastically perturbed discs are considered in Section 7, and the vertical displacements, velocities and luminosities are obtained, by numerically integrating the corresponding Langevin equation, for both the static Schwarzschild and the rotating Kerr geometries. We discuss and conclude our results in Section 8.

## 2 MOTION OF TEST PARTICLES IN STATIONARY AXISYMMETRIC GRAVITATIONAL FIELDS

The metric generated by a rotating axisymmetric compact general relativistic object can be generally given in cylindrical coordinates  $(ct, \rho, \phi, z)$  as

$$ds^2 = g_{tt} (c dt)^2 + 2g_{t\phi} c dt d\phi + g_{\phi\phi} d\phi^2 + g_{\rho\rho} (d\rho^2 + dz^2), \quad (2)$$

where all the metric functions are functions of  $\rho$  and  $z$  only. The equatorial plane is placed at  $z = 0$ . For a given time-like worldline  $x^\mu(s)$  with 4-velocity  $u^\mu = dx^\mu/ds$  and azimuthal angular velocity  $\Omega = d\phi/dt$ , the specific azimuthal angular momentum  $\tilde{L}$  of a particle of mass  $m$ , given by

$$\tilde{L} = u_\phi = u^t (g_{t\phi} + g_{\phi\phi} \Omega/c), \quad (3)$$

and the specific energy  $\tilde{E}$ , given by

$$\tilde{E} = -u_t = -u^t (g_{tt} + g_{t\phi} \Omega/c), \quad (4)$$

are constants of motion. The 4-velocity satisfies the condition  $u_\mu u^\mu = -1$ . In a stationary axisymmetric gravitational field, the most important types of worldlines correspond to spatially circular orbits, given by  $\rho = \text{constant}$ ,  $z = \text{constant}$  and  $\Omega = \text{constant}$ . In this case, the 4-velocity is given by

$$u^\mu = u^t (1, 0, \Omega/c, 0), \quad (5)$$

where

$$u^t = \frac{1}{\sqrt{-g_{tt} - 2g_{t\phi}(\Omega/c) - g_{\phi\phi}(\Omega/c)^2}} = \left( \tilde{E} - \frac{\Omega}{c} \tilde{L} \right)^{-1}. \quad (6)$$

For circular orbits, the 4-acceleration  $a_\mu = -(1/2)g_{\alpha\beta,\mu} u^\alpha u^\beta$  of the test particles has only a radial component,

$$\begin{aligned} a_\rho &= -\frac{1}{2} g_{\alpha\beta,\rho} u^\alpha u^\beta \\ &= -\left( \frac{1}{2} g_{tt,\rho} + g_{t\phi,\rho} \frac{\Omega}{c} + \frac{1}{2} g_{\phi\phi,\rho} \frac{\Omega^2}{c^2} \right) (u^t)^2. \end{aligned} \quad (7)$$

The radial component of the force  $a_\rho$  vanishes for two particular values  $\Omega_\pm$  of the angular velocity, given by

$$\frac{\Omega_\pm}{c} = \frac{-g_{t\phi,\rho} \pm \sqrt{g_{t\phi,\rho}^2 - g_{tt,\rho} g_{\phi\phi,\rho}}}{g_{\phi\phi,\rho}}. \quad (8)$$

## 3 STOCHASTIC PERTURBATION OF AN EQUATORIAL ORBIT

In the absence of any external perturbation, the particles in the disc move along the geodesic lines given by

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \quad (9)$$

where  $\Gamma_{\alpha\beta}^{\mu}$  are the Christoffel symbols associated with the metric. If the disc is perturbed, a point particle in the nearby position  $x'^{\mu}$  must also satisfy a generalized geodesic equation

$$\frac{d^2 x'^{\mu}}{ds^2} + \Gamma_{\alpha\beta}^{\mu} \frac{dx'^{\alpha}}{ds} \frac{dx'^{\beta}}{ds} = \frac{1}{m} f^{\mu}, \quad (10)$$

where  $f^{\mu}$  is the external, non-gravitational force acting on the particle. The coordinates  $x'^{\mu}$  are given by  $x'^{\mu} = x^{\mu} + \delta x^{\mu}$ , where  $\delta x^{\mu}$  is a small quantity. Thus, in the first approximation, we obtain for the Christoffel symbols

$$\Gamma_{\alpha\beta}^{\mu}(x + \delta x) = \Gamma_{\alpha\beta}^{\mu}(x) + \Gamma_{\alpha\beta,\lambda}^{\mu} \delta x^{\lambda}, \quad (11)$$

where  $\Gamma_{\alpha\beta,\lambda}^{\mu} = \partial \Gamma_{\alpha\beta}^{\mu} / \partial x^{\lambda}$ . Therefore, the equation of motion for  $\delta x^{\mu}$  becomes

$$\frac{d^2 \delta x^{\mu}}{ds^2} + 2\Gamma_{\alpha\beta}^{\mu} \frac{dx^{\alpha}}{ds} \frac{d\delta x^{\beta}}{ds} + \Gamma_{\alpha\beta,\lambda}^{\mu} \frac{dx^{\alpha}}{ds} \frac{dx^{\beta}}{ds} \delta x^{\lambda} = f^{\mu}. \quad (12)$$

If  $f^{\mu} = 0$  we re-obtain the equation of perturbation of the geodesic line introduced in Shirokov (1973). In the following, we introduce the 4-velocity of the perturbed motion as  $\delta V^{\mu} = d\delta x^{\mu}/ds$ .  $\delta V^{\mu}$  satisfies the condition  $u_{\mu} \delta V^{\mu} = 0$ . We consider that the particles in the disc are in contact with an isotropic and homogeneous heat bath. The interaction of the particles with the bath is described by a friction force, and a random force. With respect to a comoving observer frame, the heat bath has a non-vanishing average 4-velocity  $U^{\mu}$ .

In the non-relativistic case, the friction force is given by  $f_{fr}^i = -m\nu v^i$ , where  $\nu$  is the friction coefficient and  $v^i$  are the components of the non-relativistic velocity. The relativistic generalization of the friction force requires introduction of the friction tensor  $\nu_{\alpha}^{\mu}$ , given by Dunkel & Hänggi (2005a,b, 2009):

$$\nu_{\alpha}^{\mu} = \nu m (\delta_{\alpha}^{\mu} + \delta V_{\alpha} \delta V^{\mu}). \quad (13)$$

The friction force can then be expressed as

$$\begin{aligned} f_{fr}^{\mu} &= -\nu_{\alpha}^{\mu} (\delta V^{\alpha} - U^{\alpha}) \\ &= -\nu m (\delta_{\alpha}^{\mu} + \delta V_{\alpha} \delta V^{\mu}) (\delta V^{\alpha} - U^{\alpha}). \end{aligned} \quad (14)$$

Let us now introduce a Gaussian stochastic vector field  $m\xi^{\mu}[g; x]$ , where  $g$  is the determinant of the metric tensor, defined by the following correlators:  $\langle \xi^{\mu}[g; x] \rangle = 0$  and  $\langle \xi^{\mu}[g; x] \xi^{\nu}[g; y] \rangle = D^{\mu\nu}[g; x; y]$ , where  $D^{\mu\nu}[g; x; y]$  is the noise kernel tensor. From a physical point of view,  $m\xi^{\mu}[g; x]$  represents the stochastic force generated by the interaction of the particles of the disc with an external heat bath. Thus, the motion of the general relativistic massive test particles in the stochastically perturbed disc can be described by the following general relativistic Langevin-type equation:

$$\begin{aligned} \frac{d\delta V^{\mu}}{ds} + 2\Gamma_{\alpha\beta}^{\mu} u^{\alpha} \delta V^{\beta} + \Gamma_{\alpha\beta,\lambda}^{\mu} u^{\alpha} u^{\beta} \delta x^{\lambda} &= \\ &= -\nu (\delta_{\alpha}^{\mu} + \delta V_{\alpha} \delta V^{\mu}) (\delta V^{\alpha} - U^{\alpha}) + \xi^{\mu}[g; x]. \end{aligned} \quad (15)$$

In the following, we will use equation (15) to analyse the physical properties of randomly fluctuating accretion discs.

#### 4 STOCHASTIC OSCILLATIONS OF ACCRETION DISCS

We assume that the heat bath is at rest, so that  $U^{\mu} = (g_{tt}^{-1/2}, 0, 0, 0)$ . Also we assume that, since the perturbations are small, the variation of the gravitational field along the  $z$ -direction can be neglected. This implies that along the equatorial plane  $g_{\mu\nu,z} = 0$ , but  $g_{\mu\nu,zz} \neq 0$ . Then from Langevin equation (32) we obtain the

equations of motion of the particles in the stochastically perturbed disc as

$$\begin{aligned} \frac{d^2 \delta t}{ds^2} + 2 \left( \Gamma_{t\rho}^t + \Gamma_{\phi\rho}^t \frac{\Omega}{c} \right) u^t \frac{d\delta\rho}{ds} &= \\ &= -\nu (\delta_{\alpha}^t + \delta V_{\alpha} \delta V^t) (\delta V^{\alpha} - U^{\alpha}) + \xi^t[g; x], \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{d^2 \delta\rho}{ds^2} + 2 \left( \Gamma_{t\rho}^{\rho} + \Gamma_{t\phi}^{\rho} \frac{\Omega}{c} \right) u^t \frac{d\delta\rho}{ds} &+ \\ 2 \left( \Gamma_{t\phi}^{\rho} + \Gamma_{\phi\phi}^{\rho} \frac{\Omega}{c} \right) u^t \frac{d\delta\phi}{ds} &+ \\ \left[ \Gamma_{tt,\rho}^{\rho} + 2\Gamma_{t\phi,\rho}^{\rho} \frac{\Omega}{c} + \Gamma_{\phi\phi,\rho}^{\rho} \left( \frac{\Omega}{c} \right)^2 \right] (u^t)^2 \delta\rho &= \\ &= -\nu (\delta_{\alpha}^{\rho} + \delta V_{\alpha} \delta V^{\rho}) (\delta V^{\alpha} + \xi^{\rho}[g; x]), \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{d^2 \delta\phi}{ds^2} + 2 \left( \Gamma_{t\rho}^{\phi} + \Gamma_{\phi\rho}^{\phi} \frac{\Omega}{c} \right) u^t \frac{d\delta\rho}{ds} &= \\ &= -\nu (\delta_{\alpha}^{\phi} + \delta V_{\alpha} \delta V^{\phi}) (\delta V^{\alpha} + \xi^{\phi}[g; x]), \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{d^2 \delta z}{ds^2} + \left[ \Gamma_{tt,z}^z + 2\Gamma_{t\phi,z}^z \frac{\Omega}{c} + \Gamma_{\phi\phi,z}^z \left( \frac{\Omega}{c} \right)^2 \right] (u^t)^2 \delta z &= \\ &= -\nu (\delta_{\alpha}^z + \delta V_{\alpha} \delta V^z) (\delta V^{\alpha} + \xi^z[g; x]). \end{aligned} \quad (19)$$

In the absence of the friction force and the stochastic force generated by the interaction between a disc and a heat bath, we re-obtain the equations for the free oscillations of accretion discs (Semerak & Zacek 2000).

#### 5 ENERGY TRANSFER IN STOCHASTICALLY OSCILLATING DISCS

In the following, we will use the Eckart model for dissipative processes (Weinberg 1972), and we will choose a frame defined by the family of observers moving with the normalized 4-velocity  $v^{\mu} = n^{\mu}/n$  parallel to the oscillating matter fluid of the disc, where  $n^2 = -g_{\mu\nu} n^{\mu} n^{\nu}$ .

We assume that the stochastically oscillating disc consists of at least one fluid, whose particle number density flux  $n^{\mu} = n v^{\mu}$  is conserved,  $n^{\mu}_{;\mu} = 0$ , where  $;$  denotes the covariant derivative with respect to the metric, and  $n$  is the particle number density. Let us now introduce a Gaussian stochastic tensor field  $\xi_{[\mu\nu]}[g; x]$  defined by the correlators  $\langle \xi_{\mu\nu}[g; x] \rangle = 0$  and  $\langle \xi_{\alpha\beta}[g; x] \xi_{\mu\nu}[g; y] \rangle = N_{\alpha\beta\mu\nu}[g; x; y]$ , where  $\langle \rangle$  means statistical average (Hu & Verdaguer 2008). The symmetry and positive semi-definite property of the noise kernel guarantees that the stochastic field tensor  $\xi_{\mu\nu}[g; x]$ , or  $\xi_{\mu\nu}$  for short, is well defined. We assume that this general relativistic stochastic tensor describes the fluctuations of the energy-momentum tensor  $\delta T^{\mu\nu}$  of the matter in the disc due to the interaction with the thermal bath. Hence,  $\delta T^{\mu\nu}$  can be written as

$$\delta T^{\mu\nu} = \rho v^{\mu} v^{\nu} + 2q^{(\mu} v^{\nu)} + \pi^{\mu\nu} + \xi^{\mu\nu}, \quad (20)$$

where  $\rho$  is the energy density,  $q^{\mu}$  is the transverse momentum and  $\pi^{\mu\nu}$  is the anisotropic stress tensor, as measured by an observer moving with the particle flux.  $q^{\mu}$  and  $\pi^{\mu\nu}$  satisfy the relations  $q^{\mu} v_{\mu} = 0$  and  $\pi^{\mu\nu} v_{\mu} = 0$ , respectively. Within the framework of this physical interpretation, we can define the internal energy  $\epsilon$  through the relation  $n(\epsilon + \epsilon_0) = \rho$ , where  $\epsilon_0$  is an arbitrary constant (Hawking & Ellis 1973). The energy-momentum tensor is

covariantly conserved, so that  $\delta T_{;\mu}^{\mu\nu} = 0$ . By contracting  $\delta T^{\mu\nu}$  with the observer's 4-velocity  $v_\mu$  we obtain

$$(v_\mu \delta T^{\mu\nu})_{;\nu} = v_{\mu;\nu} \delta T^{\mu\nu}. \quad (21)$$

For the first term in equation (21) we obtain

$$v_\mu \delta T^{\mu\nu} = -\rho v^\nu - q^\nu + v_\mu \xi^{\mu\nu}. \quad (22)$$

By using the definition of the internal energy and the conservation of the particle number density flux, we obtain

$$(v_\mu \delta T^{\mu\nu})_{;\nu} = -n\dot{\epsilon} - q_{;\mu}^\mu + v_{\mu;\nu} \xi^{\mu\nu} + v_\mu \xi_{;\nu}^{\mu\nu}, \quad (23)$$

where we have denoted  $\dot{\epsilon} = \epsilon_{;\nu} v^\nu$ . The right-hand side of equation (21) can be rewritten as

$$v_{\mu;\nu} \delta T^{\mu\nu} = q^\mu \dot{v}_\mu + v_{\mu;\nu} \pi^{\mu\nu} + v_{\mu;\nu} \xi^{\mu\nu}. \quad (24)$$

Thus, the local form of the energy balance of the stochastically oscillating disc is given, from the fluid observer's point of view, by

$$n\dot{\epsilon} + q_{;\mu}^\mu + q^\mu \dot{v}_\mu + v_{\mu;\nu} \pi^{\mu\nu} = v_\mu \xi_{;\nu}^{\mu\nu}. \quad (25)$$

When the stochastic term in the energy-momentum tensor is ignored, and by assuming that in this case  $\delta T_{\mu\nu}$  takes the form  $\delta T_{\mu\nu} = (\rho + p)v^\mu v^\nu + p g^{\mu\nu}$ , we can define the hydrostatic pressure  $p$  of the fluid as the trace of the stress tensor  $\pi^{\mu\nu}$ ,  $p = (1/3)\pi_\mu^\mu$ . This allows us to define the viscous stress tensor  $P^{\mu\nu}$  as (Weinberg 1972)

$$P^{\mu\nu} = -\pi^{\mu\nu} + p h^{\mu\nu}, \quad (26)$$

where  $h^{\mu\nu}$  is the orthogonal projection operator to the observer's 4-velocity defined as  $h^{\mu\nu} = g^{\mu\nu} + v^\mu v^\nu$ . The viscous stress tensor can be obtained in a general form as

$$P^{\mu\nu} = \lambda [\sigma^{\mu\nu} + 2\dot{v}^{(\mu} v^{\nu)}], \quad (27)$$

where  $\lambda$  is the viscosity coefficient and  $\sigma_{\mu\nu} = v_{(\mu;\nu)} - (1/3)v_{;\alpha}^\alpha h_{\mu\nu}$ . By defining the invariant specific volume  $v$  as the inverse of the particle number density  $n$ , it follows that  $h^{\mu\nu} v_{\mu;\nu} = v_{;\mu}^\mu = n\dot{v}$ . Then for the equation of the local energy balance, we obtain

$$n(\dot{\epsilon} + p\dot{v}) + q_{;\mu}^\mu + q^\mu \dot{v}_\mu - v_{\mu;\nu} P^{\mu\nu} = v_\mu \xi_{;\nu}^{\mu\nu}. \quad (28)$$

As for the transverse momentum  $q^\mu$  it can be obtained from the relativistic analogue of Fourier's law (Weinberg 1972), namely

$$q^\mu = -\kappa h^{\mu\nu} (T_{;\nu} + T\dot{v}_\nu), \quad (29)$$

where  $\kappa$  is the thermal conductivity coefficient and  $T$  is the equilibrium temperature of the system. Equation (28) is analogous to the non-relativistic energy balance equation in the presence of a stochastic force  $\xi$ , given by

$$n \frac{d\epsilon}{dt} + \nabla \cdot \mathbf{q} - (\mathbf{\Pi} \cdot \nabla) \cdot \mathbf{v} = \mathbf{v} \cdot \boldsymbol{\xi}, \quad (30)$$

where  $\epsilon$  is the internal energy,  $\mathbf{q}$  is the heat flux,  $\mathbf{\Pi}$  is the total stress tensor, and  $\mathbf{v}$  is the fluid's 3-velocity. In equation (28) only the term containing the 4-acceleration  $\dot{v}_\mu$  does not have a Newtonian counterpart. When  $\xi \equiv 0$ , equation (30) reduces to the standard energy balance equation in non-relativistic fluid mechanics (Landau & Lifshitz 1987).

## 6 EQUATION OF MOTION OF VERTICALLY PERTURBED ACCRETION DISCS

In the following, we will consider only the vertical oscillations of the disc, which, for an arbitrary axisymmetric metric, are described

by the equation

$$\begin{aligned} \frac{d^2 \delta z}{ds^2} + \left[ \Gamma_{tt,z}^z + 2\Gamma_{t\phi,z}^z \frac{\Omega}{c} + \Gamma_{\phi\phi,z}^z \left( \frac{\Omega}{c} \right)^2 \right] (u^t)^2 \delta z \\ = -v \left( \delta_\alpha^z + \delta V_\alpha \delta V^\alpha \right) \delta V^\alpha + \xi^z [g; z]. \end{aligned} \quad (31)$$

Since  $\delta V^\mu$  is a small quantity, the equation of motion can be further simplified to

$$\frac{d^2 \delta z}{ds^2} + v \frac{d\delta z}{ds} + \omega_\perp^2 \delta z = \xi^z [g; z], \quad (32)$$

where

$$\omega_\perp^2 = \left[ \Gamma_{tt,z}^z + 2\Gamma_{t\phi,z}^z \frac{\Omega}{c} + \Gamma_{\phi\phi,z}^z \left( \frac{\Omega}{c} \right)^2 \right] (u^t)^2. \quad (33)$$

The angular frequency with respect to the radial infinity is obtained as

$$\Omega_\perp^2 = \omega_\perp^2 / (u^t)^2 = \Gamma_{tt,z}^z + 2\Gamma_{t\phi,z}^z \frac{\Omega}{c} + \Gamma_{\phi\phi,z}^z \left( \frac{\Omega}{c} \right)^2. \quad (34)$$

In the following, we consider the vertical oscillations of the discs around black holes described by the Schwarzschild and the Kerr geometries.

### 6.1 Disc oscillation frequencies in the Schwarzschild geometry

The general form of a static axisymmetric metric can be given as (Binni et al. 2005)

$$ds^2 = -e^{2U} c^2 dt^2 + e^{-2U} [e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2]. \quad (35)$$

For the Schwarzschild solution,

$$U = \frac{1}{2} \ln \frac{\sqrt{\rho^2 + (M+z)^2} + \sqrt{\rho^2 + (M-z)^2} - 2M}{\sqrt{\rho^2 + (M+z)^2} + \sqrt{\rho^2 + (M-z)^2} + 2M}, \quad (36)$$

and

$$\gamma = \frac{1}{2} \ln \frac{[\sqrt{(M+z)^2 + \rho^2} + \sqrt{(M-z)^2 + \rho^2}]^2 - 4M^2}{4\sqrt{(M+z)^2 + \rho^2} \sqrt{(M-z)^2 + \rho^2}}, \quad (37)$$

where  $M$  is the mass of the central compact object in geometrical units (Binni et al. 2005). The usual Schwarzschild line element in  $(ct, r, \theta, \phi)$  coordinates is recovered by carrying out the transformations  $\rho = \sqrt{r^2 - 2Mr} \sin \theta$  and  $z = (r - M) \cos \theta$ .

For an accretion disc in the Schwarzschild geometry, the radial frequency at infinity and the proper frequency are given by

$$\Omega_\perp^2(\rho, z, M) = \frac{e^{4U-2\gamma}}{1 - \rho U_{,\rho}} U_{,zz} \quad (38)$$

and

$$\omega_\perp^2(\rho, z, M) = \frac{e^{2U-2\gamma}}{1 - 2\rho U_{,\rho}} U_{,zz}, \quad (39)$$

respectively. Estimating  $\Omega_\perp^2$  in the equatorial plane at  $z = 0$  gives  $\Omega_\perp^2|_{z=0} = M / (M + \sqrt{M^2 + \rho^2})^3$ , while for  $\omega_\perp^2$  we obtain

$$\omega_\perp^2(M, \rho) = \frac{M}{\rho^2 \sqrt{M^2 + \rho^2} - 2M^2 (M + \sqrt{M^2 + \rho^2})}. \quad (40)$$

## 6.2 Disc oscillation frequencies in the Kerr geometry

The line element of a stationary axisymmetric space–time is given by the Lewis–Papapetrou metric (Binni et al. 2005)

$$ds^2 = e^{2(\gamma-\psi)} (d\rho^2 + dz^2) + \rho^2 e^{-2\psi} d\phi^2 - e^{2\psi} (c dt - \omega d\phi)^2, \quad (41)$$

where  $\gamma$  and  $\psi$  are functions of  $z$  and  $\rho$  only. The metric functions generating the Kerr black hole solution in axisymmetric form are given by

$$\psi = \frac{1}{2} \ln \frac{(R_+ + R_-)^2 - 4M^2 + (a^2/\sigma^2)(R_+ - R_-)^2}{(R_+ + R_- + 2M)^2 + (a^2/\sigma^2)(R_+ - R_-)^2}, \quad (42)$$

$$\gamma = \frac{1}{2} \ln \frac{(R_+ + R_-)^2 - 4M^2 + (a^2/\sigma^2)(R_+ - R_-)^2}{4R_+ R_-}, \quad (43)$$

$$\omega = -\frac{aM}{\sigma^2} \frac{(R_+ + R_- + 2M)[(R_+ - R_-)^2 - 4\sigma^2]}{(R_+ + R_-)^2 - 4M^2 + (a^2/\sigma^2)(R_+ - R_-)^2}, \quad (44)$$

where  $M$  and  $a$  are the mass and the specific angular momentum of the compact object,  $R_{\pm} = \sqrt{\rho^2 + (z \pm \sigma)^2}$  and  $\sigma = \sqrt{M^2 - a^2}$ . The usual form of the Kerr line element in Boyer–Lindquist coordinates  $(t, r, \theta, \phi)$  is recovered by performing the coordinate transformations  $\rho = \sqrt{r^2 - 2Mr + a^2} \sin \theta$  and  $z = (r - M) \cos \theta$ .

The frequency of the disc oscillations in the Kerr geometry is given by

$$\omega_{\perp}^2(\rho, M, a) = \frac{M(\varpi_1 a + \varpi_3)}{\varpi_2 a + \varpi_4}, \quad (45)$$

where we have denoted

$$\varpi_1 = 4M^{3/2} R_{\sigma} \delta_{+}^{13/2} \times [8M(2M + R_{\sigma})\delta_{+} + 2(5M + 3R_{\sigma})\rho^2], \quad (46)$$

$$\varpi_2 = -8M^{3/2} R_{\sigma} \delta_{+}^{19/2} (4M\delta_{+} + 2R_{\sigma}\delta_{+} + \rho^2), \quad (47)$$

$$\varpi_3 = 4R_{\sigma} \delta_{+}^6 [2(8M^3 - 3MR_{\sigma}^2 - R_{\sigma}^3)\delta_{+}^2 + 4R_{\sigma} \delta_{+}^6 [3(6M^2 + 3MR_{\sigma} + R_{\sigma}^2)\delta_{+}\rho^2 + 3M\rho^4] \quad (48)$$

and

$$\varpi_4 = -4R_{\sigma} \delta_{+}^{10} \times [(8M^3 - 3MR_{\sigma}^2 - R_{\sigma}^3)\delta_{+} + 3M(2M + R_{\sigma})\rho^2], \quad (49)$$

with  $\delta_{\pm} = \sqrt{\rho^2 + \sigma^2} \pm M$  and  $R_{\sigma} = \sqrt{\rho^2 + \sigma^2}$ . In the limit  $a = 0$ ,  $R_{\sigma} = \sqrt{M^2 + \rho^2}$  and  $\delta_{\pm} = R_{\sigma} \pm M$ , and after some algebraic manipulation, we recover equation (40) giving the oscillation frequency of the Schwarzschild disc.

## 7 VERTICAL STOCHASTIC OSCILLATIONS OF PERTURBED ACCRETION DISCS

By assuming that the velocity of the perturbed disc oscillations is small, we can approximate  $ds$  as  $ds \approx c dt$ , where  $t$  is the time-coordinate. We also assume that the external perturbative force is a function of time only, and neglect its possible spatial coordinate dependence. In this case, the equation of motion of the disc in contact with a heat bath follows from equation (32) and is given by

$$\frac{d^2 \delta z}{dt^2} + cv \frac{d\delta z}{dt} + c^2 \omega_{\perp}^2(\rho, M, a) \delta z = c^2 \xi^z(t), \quad (50)$$

where the oscillation frequency of the disc is given by equation (40) in the Schwarzschild case, and by equation (45) in the Kerr case. The random force  $\xi^z(t)$  has a zero mean and a variance  $\langle \xi^z(t) \xi^z(t') \rangle = (D/c^2) \delta(t - t')$ , where  $\delta(t - t')$  is the Dirac delta function and  $D \geq 0$ . By introducing the velocity  $\delta V^z = d\delta z/dt$  of the vertical disc oscillation, equation (50) can be written in the form

$$d\delta z = \delta V^z dt, \quad (51)$$

$$d\delta V^z = -cv \delta V^z dt - c^2 \omega_{\perp}^2(\rho, M) \delta z dt + D^{1/2} dW(t), \quad (52)$$

where  $W(t)$  is a collection of standard Wiener processes.

A Wiener process,  $W(t) \geq 0$ , is a one-parameter family of Gaussian random variables, with expectations zero and covariances  $E(W(s)W(t)) = \min\{s, t\}$ . Because  $W(t)$  are all Gaussian, this information suffices to determine joint probabilities. Alternatively,  $W(t)$  may be viewed as a ‘random’ continuous function with  $W(0) = 0$ . A Wiener process may be generated at consecutive grid points  $t_n$  by  $W(0) = 0$ ,  $W(t_n) = W(t_{n-1}) + (t_n - t_{n-1})Z_n$ , and  $\{Z_n\}$  is a set of independent standard Gaussian random variables (with mean 0 and variance 1) (Skeel & Izaguirre 2002).

In order to numerically integrate equations (51) and (52), we consider the Brünger–Brooks–Karplus (BBK) scheme (Brünger, Brooks & Karplus 1984). If the time-step is taken as  $\Delta t$ , we denote  $\delta z_n \approx \delta z(n\Delta t)$ , and we define  $f_n = -c^2 \omega_{\perp}^2(\rho, M, a) \delta z_n \Delta t^2$ . The recurrence relation for the BBK integrator is

$$\delta z_{n+1} = \frac{2}{1 + g/2} \delta z_n - \frac{1 - g/2}{1 + g/2} \delta z_{n-1} + \frac{1}{1 + g/2} f_n + \frac{1}{1 + g/2} (D\Delta t^3)^{1/2} Z_n, \quad (53)$$

where we have denoted  $g = cv\Delta t$ . The starting procedure of the BBK integrator is given by

$$\delta z_1 = \delta z_0 + \left(1 - \frac{g}{2}\right) V_0 \Delta t + \frac{1}{2} f_0 + \left(\frac{D\Delta t^3}{4}\right)^{1/2} Z_0, \quad (54)$$

where  $\delta z_0$  and  $V_0$  are the initial displacement and velocity, respectively. The velocity  $\delta V^z$  can be obtained as

$$\delta V_n^z = \frac{\delta z_n - \delta z_{n-1}}{\Delta t}. \quad (55)$$

By defining

$$E = \frac{1}{2} \left( \frac{d\delta z}{dt} \right)^2 + \frac{1}{2} c^2 \omega_{\perp}^2(\rho, M, a) \delta z^2 \quad (56)$$

as the total energy per unit mass of the oscillating disc, the luminosity per unit mass  $L$  of the disc, representing the energy lost by the disc due to viscous dissipation and due to the presence of the random force, is given by

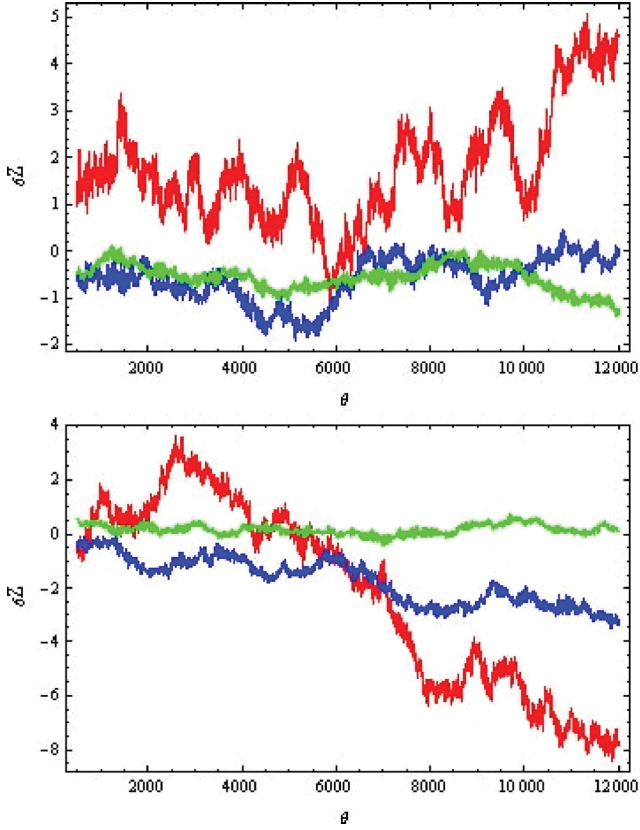
$$L = -\frac{dE}{dt} = cv \left( \frac{d\delta z}{dt} \right)^2 - c^2 \frac{d\delta z}{dt} \xi^z(t). \quad (57)$$

### 7.1 The Schwarzschild stochastic disc

In the case of an accretion disc in the static Schwarzschild geometry of a black hole, the proper frequency of the disc oscillations  $\omega_{\perp}^2(M, \rho)$  is given by equation (40). By expressing the radial distance  $\rho$  in terms of the gravitational radius  $M$  so that  $\rho = nM$ ,  $n = \text{constant} \geq 6$ , we obtain

$$\omega_{\perp}^2 = \frac{1}{M^2 \left[ (n^2 - 2) \sqrt{1 + n^2} - 2 \right]}. \quad (58)$$





**Figure 1.** Variation of the dimensionless coordinate  $\delta Z$  of the stochastically oscillating Schwarzschild disc as a function of  $\theta$  for a central object with mass  $M = M^{10} M_{\odot}$  and  $\zeta = 100$  (red upper curve),  $\zeta = 2.5 \times 100$  (blue lower curve) and  $\zeta = 5 \times 100$  (green middle curve). In the upper panel  $n = 7$ , while in the lower panel  $n \rightarrow \infty$ . (Colour versions of all Figures available in the online version of this article.)

By introducing a set of dimensionless variables  $(\theta, \delta Z)$ , defined as  $t = (M/c)\theta$  and  $\delta z = M\delta Z$ , and by denoting the product  $vM$  as  $\zeta$ , the equation of motion of the stochastic oscillating disc becomes

$$\frac{d^2 \delta Z}{d\theta^2} + \zeta \frac{d\delta Z}{d\theta} + \frac{1}{(n^2 - 2)\sqrt{1 + n^2 - 2}} \delta Z = \eta^z(\theta), \quad (59)$$

where  $\eta^z(\theta) = M\xi^z(t)$ .

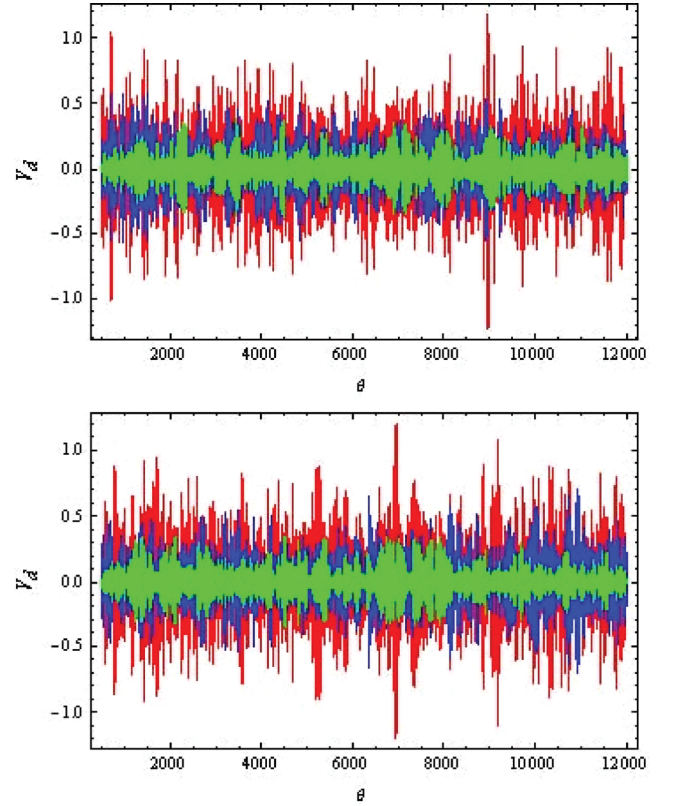
The dimensionless luminosity  $L_{\theta}$  can be written as

$$L_{\theta} = \zeta \left( \frac{d\delta Z}{d\theta} \right)^2 - \frac{d\delta Z}{d\theta} \eta^z(\theta), \quad (60)$$

where  $L_{\theta} = ML/c^3$ .

The variations of the dimensionless coordinate,  $\delta Z$ , of the stochastic velocity of the disc and of the luminosity of the stochastically perturbed disc are represented, for a central object with mass  $M = 10^{10} M_{\odot}$ , for  $n = 7$  and  $n \rightarrow \infty$ , and for several values of  $\zeta$ , in Figs 1–3.

In order to describe the stochastic characteristics of the physical parameters of the oscillating disc from a quantitative point of view, we compute the power spectral density (PSD) of the luminosity (Vaughan 2010, for details see the appendix). The numerical values of the slopes of the PSD give an insight into the nature of the mechanism leading to the observed variability. The computations of the PSD have been done by using the `R` software and the assumption  $\mathcal{H} : P(f) = \beta f^{-\alpha}$ , where  $\alpha$  and  $\beta$  are constants, which was applied to the time-series obtained by using the BBK integrator. The results for the Schwarzschild case are shown in Fig. 4. In the



**Figure 2.** Variation of the dimensionless velocity  $V_d = d\delta Z/d\theta$  of the stochastically oscillating Schwarzschild disc as a function of  $\theta$  for a central object with mass  $M = M^{10} M_{\odot}$  and for  $\zeta = 100$  (red upper curve),  $\zeta = 2.5 \times 100$  (blue middle curve) and  $\zeta = 5 \times 100$  (green lower curve). In the upper panel  $n = 7$ , while in the lower panel  $n \rightarrow \infty$ .

case considered in this paper, the luminosity is determined by the evolution, according to a Brownian-type equation of motion, of a volume element of the matter in the disc. For such a situation, we expect a PSD slope of  $-2$ , and this result in fact represents a consistency check of the stochastic simulations.

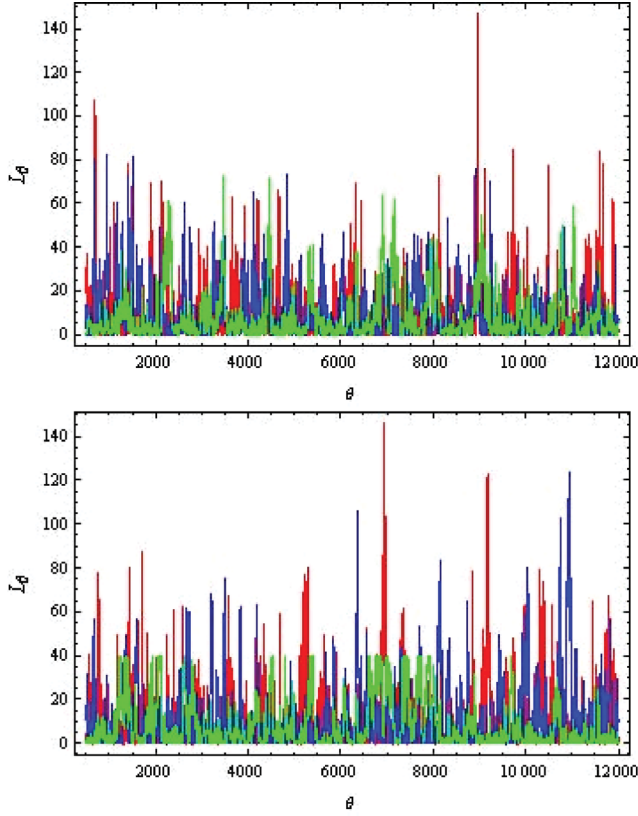
## 7.2 The Kerr stochastic disc

By denoting  $\rho = nM$ ,  $a = kM$  and  $\delta = M\sqrt{n^2 + 1 - k^2}$ , with  $n > 0$  and  $k \in [0, 1]$ , the oscillation frequency of the Kerr disc, given by equation (45), can be represented as

$$\omega_{\perp}^2 = \frac{\omega_1^2(n, k)}{M^2 \omega_2^2(n, k)}, \quad (61)$$

where

$$\begin{aligned} \omega_1^2(n, k) = & 3n^4 + n^2\sqrt{1 + \delta} \\ & \times \left[ 2k(5 + 3\delta) + 3\sqrt{1 + \delta}(6 + 3\delta + \delta^2) \right] \\ & - 2(1 + \delta)^{3/2} \times \left[ -4k(2 + \delta) \right. \\ & \left. + \sqrt{1 + \delta}(-8 + 3\delta^2 + \delta^3) \right] \end{aligned} \quad (62)$$



**Figure 3.** Dimensionless luminosity  $L_\theta$  of the stochastically oscillating Schwarzschild disc for a central object with mass  $M = M^{10} M_\odot$  and  $\zeta = 100$  (red upper curve),  $\zeta = 2.5 \times 100$  (blue middle curve) and  $\zeta = 5 \times 100$  (green lower curve). In the upper panel  $n = 7$ , while in the lower panel  $n \rightarrow \infty$ .

and

$$\omega_2^2(n, k) = (1 + \delta) [n^2 - \delta(1 + \delta)(3 + \delta)]^2 \times \left\{ \delta - \frac{[k + (1 + \delta)^{3/2}]^2 [4(1 + \delta) + n^2(3 + \delta)]}{[n^2 - \delta(1 + \delta)(3 + \delta)]^2} + \frac{4k[k + (1 + \delta)^{3/2}]}{n^2 - \delta(1 + \delta)(3 + \delta)} - 1 \right\}. \quad (63)$$

Then the vertical dimensionless equation of motion of the stochastically oscillating Kerr disc is given by

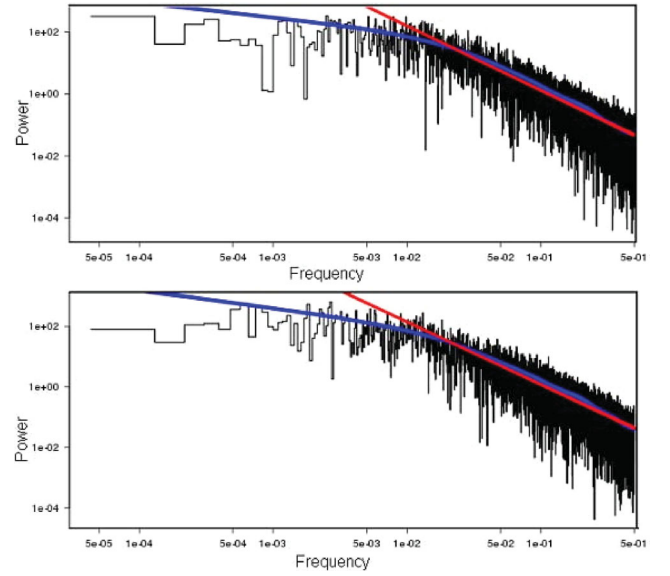
$$\frac{d^2 \delta Z}{d\theta^2} + \zeta \frac{d\delta Z}{d\theta} + \frac{\omega_1^2(n, k)}{\omega_2^2(n, k)} \delta Z = \eta^z(\theta). \quad (64)$$

In Figs 5–7 we have represented the variations of the dimensionless coordinate  $\delta Z$ , of velocity  $V_d = d\delta Z/d\theta$ , and of the luminosity for a stochastically oscillating Kerr disc for a rotating massive central object with mass  $M = 10^{10} M_\odot$  and  $a = 0.9M$ , for  $n = 7$ , and for different values of  $\zeta$ .

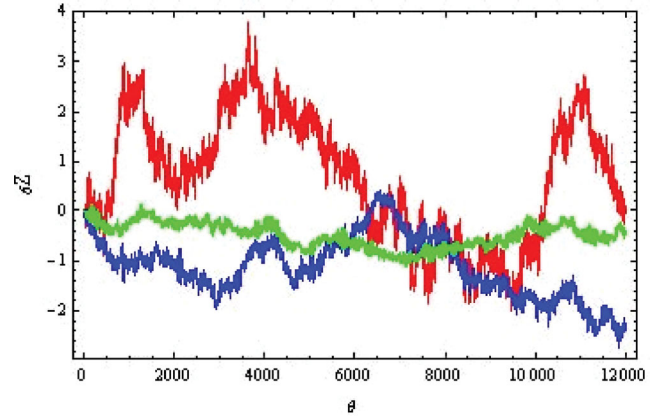
In Fig. 8 we show the PSD of the luminosity of the stochastically oscillating disc around a Kerr black hole with mass  $M = 10^{10} M_\odot$  and  $a = 0.9M$ .

## 8 DISCUSSION AND FINAL REMARKS

In this paper, we have introduced a mathematical model for the description of the perturbations of a thin accretion disc in contact

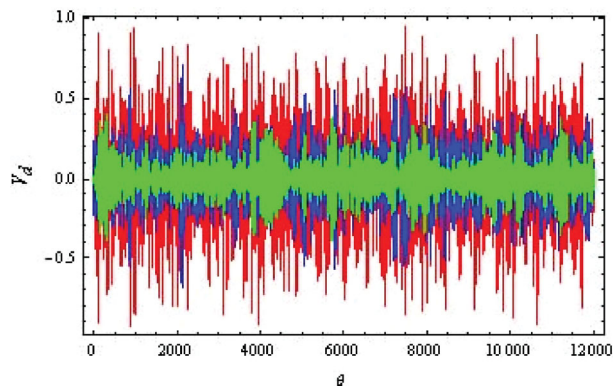


**Figure 4.** PSD of the luminosity of the stochastically oscillating Schwarzschild disc for a central object of mass  $M = M^{10} M_\odot$  and for  $\zeta = 100$ . In the upper panel  $n = 7$  and  $\alpha = 2.076$ . In the lower panel  $n \rightarrow \infty$  and  $\alpha = 2.070$ . The red lines represent the fits according to  $\mathcal{H}$ .

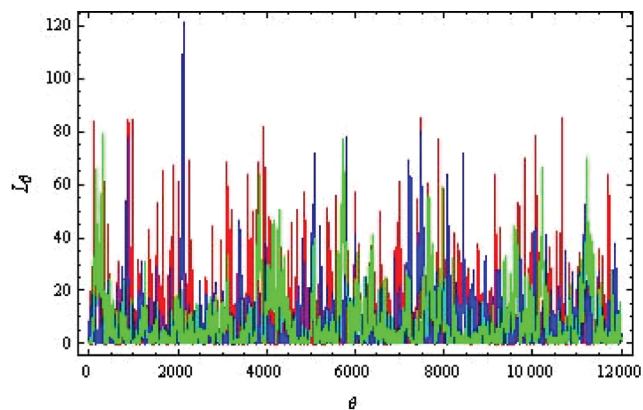


**Figure 5.** Variation of the dimensionless coordinate  $\delta Z$  of the stochastically oscillating disc as a function of  $\theta$  for a rotating Kerr black hole with mass  $M = M^{10} M_\odot$  and  $a = 0.9M$ , for  $n = 7$ , and for different values of  $\zeta$ :  $\zeta = 100$  (red upper curve),  $\zeta = 2.5 \times 100$  (blue lower curve) and  $\zeta = 5 \times 100$  (green middle curve).

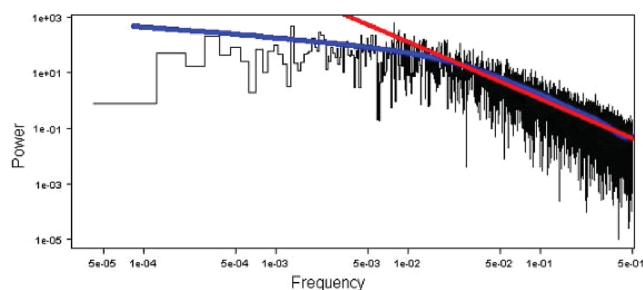
with an exterior stochastic medium (a thermal bath). As a result of the interaction between the medium and the disc, the motion of the particles in the disc has essentially a stochastic nature. To obtain the equations of motion of the perturbed disc, we have generalized the approach introduced in Shirokov (1973) and Semerak & Zacek (2000), by introducing in the equations of the perturbed geodesic lines a dissipative viscous term and a stochastic force term, respectively. The equations of motion of the fluctuating disc have been obtained in both the equatorial and the vertical planes, as well as the general stochastic energy transfer equation. We have studied in detail the vertical oscillations of the disc, and we have shown that in this case the motion is described by a standard Langevin-type differential equation for a harmonic oscillator, with a geometry-dependent oscillation frequency. By numerically integrating the equation of motion, we have obtained the vertical displacement, velocity and luminosity of the disc in the cases of both the Schwarzschild and the Kerr geometries.



**Figure 6.** Variation of the dimensionless velocity  $V_d = d\delta Z/d\theta$  of the stochastically oscillating disc as a function of  $\theta$  for a rotating Kerr black hole with mass  $M = M^{10} M_\odot$  and  $a = 0.9M$ , for  $n = 7$ , and for different values of  $\zeta$ :  $\zeta = 100$  (red upper curve),  $\zeta = 2.5 \times 100$  (blue middle curve) and  $\zeta = 5 \times 100$  (green lower curve).



**Figure 7.** Variation of the luminosity  $L_\theta$  of the stochastically oscillating disc as a function of  $\theta$  for a rotating Kerr black hole with mass  $M = M^{10} M_\odot$  and  $a = 0.9M$ , for  $n = 7$ , and for different values of  $\zeta$ :  $\zeta = 100$  (red lower curve),  $\zeta = 2.5 \times 100$  (blue middle curve) and  $\zeta = 5 \times 100$  (green upper curve).



**Figure 8.** PSD of the luminosity for a stochastically oscillating disc around a Kerr black hole with mass  $M = M^{10} M_\odot$  and  $a = 0.9M$ , for  $n = 7$  and  $\zeta = 100$ . The slope of the PSD is 2.046.

Fluctuations of accretion discs around supermassive black holes may have important astrophysical applications. The emission of Galactic black hole binaries (BHBs) and AGNs displays a significant aperiodic variability on a broad range of time-scales. The PSD of such variability is generally modelled with a power law,  $P(f) \propto f^{-\alpha}$ , where the power-law index  $\alpha$  keeps a constant value in a certain range of  $f$ , but changes between different ranges. At high frequencies, the PSDs of both BHBs and AGNs present a

steep slope with  $\alpha \sim 2$ . In contrast, below a break frequency, typically at a few Hz for BHBs, they flatten to a slope with  $\alpha \sim 1$ , representing the so-called flicker noise (King et al. 2004). However, the observed power spectra often deviate from the form  $f^{-1}$ . For example, the PSD of Cyg X-1 is well described with the form  $f^{-1}$  in the soft state, but in the hard state it exhibits the form  $f^{-1.3}$  (Gilfanov 2010). In particular, it was shown that the power-law index is around  $\alpha = 0.8$ – $1.3$  both in the soft state of BHBs and in narrow-line Seyfert 1 galaxies (Janiuk & Czerny 2007).

In the case of the present model of the disc oscillations under the effect of some stochastic forces  $\xi(t)$ , the resulting PSD is of the form  $P(f) \propto f^{-2}$ . This result can be understood qualitatively by noting that  $P(f) \propto \int \exp(ift) [\int^t \xi(t') dt'] dt \propto if^{-1} \int^t \exp(ift) \xi(t) dt$ , which gives  $P \propto f^{-2}$ , since  $\xi(t)$  is a stochastic variable. This result is generally also valid for harmonically oscillating, undamped discs, and consequently it is independent of the general relativistic corrections to the equation of motion, since these only change the oscillation frequency of the compact object–disc system. However, as pointed out previously, observations of BHBs and AGNs have shown that the spectral index is not  $\alpha = 2$ , but it is either  $\alpha \approx 1$  or, more generally,  $\alpha \in (0.8, 2)$ . Such a dispersion of the power-law index reveals that there must exist some other mechanisms, different from purely stochastic oscillations, which may be responsible for the observed value of  $\alpha$ , like, for example, hydrodynamic fluctuations, magnetohydrodynamic turbulence, magnetic flares, density fluctuations in the corona, or variations of the accretion rate, caused by small-amplitude variations in the viscosity (King et al. 2004).

In a model introduced in Takeuchi & Mineshige (1997), aperiodic X-ray fluctuations are thought to originate from instabilities of the accretion disc around a black hole. To describe these type of fluctuations a cellular automaton model was proposed (Mineshige, Ouchi & Nishimori 1994). In this model, a gas particle is randomly injected into an accretion disc surrounding a black hole. When the mass density of the disc exceeds some critical value at a certain point, an instability develops, and the accumulated matter begins to drift inwards as an avalanche, thereby emitting X-rays. Within the framework of this model, one can generate  $f^{-1}$ -like fluctuations in the X-ray luminosity, in spite of the random mass injection. A similar model can be used to explain the optical–ultraviolet ‘flickering’ variability in cataclysmic variables (Yonehara, Mineshige & Welsh 1997). Light fluctuations are produced by occasional flare-like events, and subsequent avalanche flow in the accretion disc atmospheres.

Stochastic oscillations of the accretion discs can also provide an alternative model for the explanation of the observed intraday variability in BL Lac objects, and for other similar transient events (Leung et al. 2011). The basic physical idea of this model is that the source of the intraday variability can be related to some stochastic oscillations of the disc, triggered by the interaction of the disc with the central supermassive black hole, as well as with a background cosmic environment, which perturbs the disc. To explain the observed light-curve behaviour, a model for the stochastic oscillations of the disc was developed, by taking into account the gravitational interaction with the central object, the viscous-type damping forces generated in the disc, and a stochastic component that describes the interaction with the cosmic environment. The stochastic oscillation model can reproduce the aperiodic light curves associated with transient astronomical phenomena.

Random or fluctuating phenomena can be found in many natural processes. In this paper, we have investigated the stochastic properties of the oscillating general relativistic thin accretion discs, and we have introduced an approach in which the stochastic processes



related to disc instabilities can be described in a unitary approach by a Langevin-type equation that includes the deterministic, general relativistic and random forces acting on the accretion discs. Our results can be considered only as a first step in the investigation of the stochastic properties of the general relativistic accretion discs. In order to obtain a more accurate determination of the relevant physical parameters of the discs, and to determine more exactly the possible astrophysical applications of the model, it would be important also to investigate the radial stochastic oscillations of the discs around compact general relativistic objects.

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## APPENDIX A: POWER SPECTRAL DISTRIBUTION OF STOCHASTIC PROCESSES

In statistical analysis, if  $X$  is some fluctuating quantity, with mean  $\mu$  and variance  $\sigma^2$ , then a correlation function for quantity  $X$  is defined as

$$R(\tau) = \frac{\langle (X_s - \mu)(X_{s+\tau} - \mu) \rangle}{\sigma^2},$$

where  $X_s$  is the value of  $X$  measured at time  $s$  and  $\langle \rangle$  denotes averaging over all values of  $s$ . Based on the correlation function, the PSD is defined as

$$P(f) = \int_{-\infty}^{+\infty} R(\tau) e^{-i2\pi f \tau} d\tau.$$

It is straightforward to see the importance of the PSD in terms of the ‘memory’ of a given process. For example, if  $X$  is the measured luminosity of a perturbed disc, the slope of the PSD of a time-series of  $X$  provides an insight into the degree of correlation the underlying physical processes have with themselves. The system needs additional energy to fluctuate, and this mechanism is best explained for the Brownian motion, in which case the energy is thermal. Brownian motion produces a PSD of the form  $P(f) \sim f^{-2}$ . However, assessment of the PSD slope from one single observational time-series is non-trivial. In such cases it is very helpful to use dedicated software, such as the statistical software *R* (Vaughan 2010). Basically, the input to this software consists of one time-series. Any assumption about the behaviour of the source that produced this time-series may be expressed in the form of an analytical function. For example, we want to test the assumption that the source producing the time-series behaves so as to be described by

$$\mathcal{H} : P(f) = \beta f^{-\alpha}.$$

The software will return the most probable values of the parameters  $\{\alpha, \beta\}$ , and the probability that the source indeed behaves according to this hypothesis.

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