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<td><strong>Author(s)</strong></td>
<td>Chen, CC; Liew, KM; Lim, CW; Kitipornchai, S</td>
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Vibration analysis of symmetrically laminated thick rectangular plates using the higher-order theory and $p$-Ritz method

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The free vibration analysis of symmetrically laminated thick rectangular plates is examined. The $p$-Ritz method is employed in which sets of uniquely defined polynomials are used as the admissible trial displacement and rotation functions. The energy integral expressions of the laminates are derived by incorporating the shear deformation using Reddy’s higher-order plate theory [J. Appl. Mech. ASME 51, 745–752 (1984)]. The formulation is basically applicable to rectangular laminates with any combination of free, simply supported, and clamped boundary conditions. To evaluate the validity and to demonstrate the applicability of the proposed method, a series of free vibration analyses of laminated composite plates is reported. Wherever possible, the accuracy of this analysis is validated through comparison with available results. Efforts are made to interpret the results to provide physical insight to the problem. © 1997 Acoustical Society of America.

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INTRODUCTION

The unique properties of composite laminates, including high strength and high stiffness-to-weight ratios, high fatigue resistance, high damping, and potential for tailoring, have led to serious examination of their engineering performance in areas such as vibration, buckling, and stress analyses. An understanding of the vibration behavior of composite panels has particularly attracted many researchers to the possibility of furnishing an optimal design state. Numerous works have applied a direct extension of Kirchhoff’s classical theory of plates to composite laminates. The assumptions underlying this theory, however, result in underestimation of deflection and overestimation of natural frequencies of a plate. Because of this disadvantage in the classical plate theory, numerous refined theories incorporating the transverse shear deformations have been proposed.

In existing literature, the first-order laminated plate theory was due to Yang et al.,1 which was extended to the higher-order laminated plate theory by Reddy.2 The first-order theory implies a conceptual paradox as the transverse shear strain does not vanish on the top and bottom surfaces. The thick laminated plate theories were developed to improve the modeling of transverse shear distribution. In addition, the higher-order theory discards the shear correction factors required in the first-order theory. A comprehensive review of the various refined plate theories for laminates proposed over the years has been summarized by Reddy and Robbins Jr.3 and Noor and Burton.4 Out of these theories, the theory used in the present study comes under the class of a single-layer displacement-based theory.2 In this theory, the three-dimensional elasticity theory is reduced to two dimensions by replacing the laminated plate with an equivalent homogenous anisotropic plate and introducing a global displacement approximation in the thickness direction. The order of approximation is with respect to the distribution of displacements in the thickness direction. Thus, this theory provides reasonably accurate solutions in predicting the global behavior of composite laminates.

During the past decades, there has been much interest in the finite element method using various laminate plate theories because of its versatility.5–7 However, sufficient numbers of discretized elements and nodes are required if there are curved boundary conditions. The $p$-Ritz method8 somewhat improves these shortcomings by using the more traditional Rayleigh–Ritz method which assumes the entire plate as a single element and eliminates the need for discretization, mesh generation, and larger degrees of freedom. Excellent results were seen on earlier works for isotropic plates and thin and thick laminates.9–12 An important feature of the Ritz method is the selection of admissible functions in the series representing the unknown functions in the displacement field. The accuracy and convergence of the solution are greatly dependent on the choice of the trial functions. In the present analysis, the $p$-Ritz functions are employed in which sets of uniquely defined polynomials are used as the admissible trial displacement and rotation functions. To account for the transverse shear effects, Reddy’s higher-order shear deformation plate theory has been integrated with the $p$-Ritz method for solving the free vibration analysis of thick laminated plates with various boundary conditions.

It should be noted that Reddy and his associates2,5 have presented results for only plates with opposite sides simply
supported. Results for plates with general boundary conditions are still unavailable, to the authors' knowledge. Therefore, this paper attempts to provide examples of rectangular plates with various boundary conditions to show the applicability and versatility of the $p$-Ritz method, without the difficulty of mesh generation and continuity conditions of other discretization methods. Details of the analytical method and formulation for the problem of free vibration of laminated plates with combinations of clamped, simply supported, and free edges are presented. The accuracy and validity of the present method is established through convergence and comparison studies with the available literature results.

I. MATHEMATICAL FORMULATION

A. Preliminary

A thick, flat laminated plate with a thickness $h$, length $a$, width $b$, and composed of $N$ orthotropic laminae oriented at angles $\theta$ is considered. The reference Cartesian coordinate system is located at the mid-plane of the laminated plate, as depicted in Fig. 1. The laminae are assumed to possess a plane of elastic symmetry parallel to the $xy$ plane and be stacked symmetrically with respect to the middle surface of the laminated plate. The vibration frequencies of the symmetric laminates subjected to a variety of edge conditions, length-to-thickness ratios, aspect ratios, degrees of orthotropy, stacking angles, and numbers of layers are to be determined.

B. Energy expressions

By applying Reddy’s higher-order shear deformation theory, the displacements of an arbitrary point of the thick laminated plate along the $x$, $y$, and $z$ axes can be represented as

\begin{align}
    u(x,y,z,t) &= u_0(x,y,t) + z \phi_x(x,y,t) \\
    v(x,y,z,t) &= v_0(x,y,t) + z \phi_y(x,y,t) \\
    w(x,y,z,t) &= w_0(x,y,t),
\end{align}

where $(u_0, v_0, w_0)$ are the displacement components of the mid-plane along the $x$, $y$, and $z$ directions, respectively. The rotations about the $x$ and $y$ directions, respectively, are $\phi_x$ and $\phi_y$.

Assuming transverse inextensibility, the strain-displacement relationship for any lamina in the Cartesian system can be expressed as

$$
\varepsilon = Lu
$$

with

$$
\varepsilon = \begin{bmatrix}
\varepsilon_x & \varepsilon_y & \gamma_{yz} & \gamma_{xz} & \gamma_{xy}
\end{bmatrix}^T,
$$

and

\begin{align}
    \varepsilon_x &= \frac{\partial u}{\partial x}, \\
    \varepsilon_y &= \frac{\partial v}{\partial y}, \\
    \gamma_{yz} &= \frac{\partial w}{\partial y}, \\
    \gamma_{xz} &= \frac{\partial w}{\partial x}, \\
    \gamma_{xy} &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.
\end{align}
In linear elasticity analysis, the strain energy for each ply is written in the form of

\[ U_k = \frac{1}{2} \int_{V_k} e_k^T \sigma_k e_k \ dV_k, \]

where \( U_k \) and \( V_k \) are respectively the strain energy and the volume of the \( k \)th lamina. Hence, the total strain energy for the entire laminated plate is

\[ U = \frac{1}{2} \sum_{k=1}^{N} \int_{V_k} \int_{h_k}^{h_k+1} \left( \sigma_{x} e_x + \sigma_{y} e_y + \sigma_{z} e_z \right) \ dz \ dA. \]

Accordingly, the total kinetic energy \( T \) associated with the vibration of laminated plate is

\[ T = \frac{h}{2} \sum_{k=1}^{N} \int_{A} \int_{V_k} \rho_k \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \ dA \]

in which \( \rho_k \) is the mass density per volume for the \( k \)th lamina.

The equivalent modulus for a multidirectional lamina is introduced:

\[ (A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) \]

where all \( B_{ij} \) and \( E_{ij} \) vanish if laminates are stacked symmetrically about the mid-plane. The total potential energy \( U \) and total kinetic energy \( T \) can be further expanded in terms of the equivalent modulus (see the Appendix). The deflection and rotation functions of the laminate mid-plane are periodic in time. Therefore, for small amplitude vibration, we can assume that

\[ u_0(x,y,t) = U(x,y) \sin \omega t, \]
\[ v_0(x,y,t) = V(x,y) \sin \omega t, \]
\[ w_0(x,y,t) = W(x,y) \sin \omega t, \]
\[ \phi_0(x,y,t) = \Theta_0(x,y) \sin \omega t, \]
\[ \phi_1(x,y,t) = \Theta_1(x,y) \sin \omega t. \]

By substituting Eqs. (11a)–(11e) into the total potential energy \( U \) and the total kinetic energy \( T \), we can obtain the maximum strain energy \( U_{\text{max}} \) and the maximum kinetic energy \( T_{\text{max}} \). The total energy functional \( \Pi \) of the plate is defined in terms of \( U_{\text{max}} \) and \( T_{\text{max}} \) as

\[ \Pi = U_{\text{max}} - T_{\text{max}}, \]

which can be minimized using the \( p \)-Ritz method to obtain the vibration frequencies.

**C. \( p \)-Ritz method**

The displacement and rotation components, \( U(x,y) \), \( V(x,y) \), \( W(x,y) \), \( \Theta_0(x,y) \), and \( \Theta_1(x,y) \), can be further simplified by using the nondimensional expressions.
TABLE I. Notations for boundary conditions.

<table>
<thead>
<tr>
<th>Boundary constraints</th>
<th>(u_n = u_s = 0)</th>
<th>(N_n = u_s = 0)</th>
<th>(N_n = u_s = 0)</th>
<th>(N_n = N_s = 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free</td>
<td>(\partial M_n / \partial x + Q_n = 0)</td>
<td>(F_1)</td>
<td>(F_2)</td>
<td>(F_3)</td>
</tr>
<tr>
<td>Simply supported</td>
<td>(w = 0)</td>
<td>(S_1)</td>
<td>(S_2)</td>
<td>(S_3)</td>
</tr>
<tr>
<td>Clamped</td>
<td>(w = 0)</td>
<td>(C_1)</td>
<td>(C_2)</td>
<td>(C_3)</td>
</tr>
</tbody>
</table>

*Here, \(n\) and \(s\) indicate the directions normal and tangential to the corresponding supporting edges.*

\[
\xi = \frac{x}{a}, \quad \eta = \frac{y}{b}. \tag{13a} \]

\[
\Theta_a(\xi, \eta) = \sum_{i=1}^{m} c_i \varphi_i^a(\xi, \eta), \tag{14a} \]

\[
\Theta_b(\xi, \eta) = \sum_{i=1}^{m} c_i \varphi_i^b(\xi, \eta), \tag{14b} \]

\[
\Theta_c(\xi, \eta) = \sum_{i=1}^{m} c_i \varphi_i^c(\xi, \eta), \tag{14c} \]

\[
\Theta_d(\xi, \eta) = \sum_{i=1}^{m} c_i \varphi_i^d(\xi, \eta), \tag{14d} \]

\[
\Theta_e(\xi, \eta) = \sum_{i=1}^{m} c_i \varphi_i^e(\xi, \eta), \tag{14e} \]

where \(\varphi_i^a, \varphi_i^b, \varphi_i^c, \varphi_i^d, \varphi_i^e\) are the so-called \(p\)-Ritz functions which are products of two-dimensional polynomials and basic functions. The associated \(c_i^a, c_i^b, c_i^c, c_i^d, c_i^e\) are the unknown coefficients. The number of terms \(m\) in the series (14a)–(14e) can be obtained by

\[
m = \frac{(p + 1)(p + 2)}{2}. \tag{15} \]

\[
\varphi_b^b = \prod_{s=1}^{4} \left[ \Gamma_s(\xi, \eta) \right]^{14a}, \tag{16} \]

where \(\Gamma_s(\xi, \eta)\) is the boundary equation of the \(s\)th supporting edge and \(\Omega_s^a\) denotes the associated basic power. The basic functions consist of products of boundary expressions of the laminated plate raised to their associated basic powers to guarantee automatic satisfaction of geometric boundary conditions. Whitney\(^{13}\) suggested that one member of each pair of the following four quantities must be prescribed along the boundary to ensure unique solutions to the governing equations

\[
u_s; N_n \quad u_s; N_{ns} \quad \frac{\partial w}{\partial n}; M_n \quad w_s; \frac{\partial M_{ns}}{\partial s} + Q_n. \tag{18} \]

TABLE II. Powers of basic functions for various combinations of boundary conditions.

<table>
<thead>
<tr>
<th>Edge (s)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha)</td>
<td>(u)</td>
<td>(v)</td>
<td>(w)</td>
<td>(\theta_u)</td>
</tr>
<tr>
<td>(S_2S_3S_4)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(C_1C_2C_3)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(C_1C_2F_F)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(C_1C_2F_F)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(S_2S_3S_4)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(S_2S_3S_4)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(C_1S_3S_4)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>(S_2C_1S_3)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(S_2C_1S_3)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(C_1C_2C_3)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>(C_1C_2C_3)</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Therefore, many types of boundary conditions are considered along the geometric and natural boundary constraints given in Eq. (18) and summarized in Table I. For examples, there are four types of simply supported conditions: \( S_1, S_2, S_3, \) and \( S_4 \). These supports have the transverse direction \( w \) constrained and varying in-plane support conditions. The most common simply supported condition is \( S_1 \) where both the normal and tangential displacements in the midplane are constrained. Another condition having physical interpretation is \( S_2 \) with transverse \( (w) \) and tangential \( (u_s) \) directions constrained which occur mostly in composite plates and sometimes termed "freely supported." 

The basic powers \( V_s \) are assigned 0, 1, or 2 depending on whether the normal, tangential, or transverse direction is constrained at the edge. Details of basic power for various combinations of boundary conditions are listed in Table II. As shown in Fig. 1, \( s = 1 \) refers to the edge at \( x = -a/2 \), and \( s = 2, 3, 4 \) correspond to the subsequent edges, going counterclockwise. In detail, the nondimensional basic functions for the rectangular laminated plate are

\[
\varphi_{\alpha}^a = (\xi - 0.5)^{\Omega_a}(\eta - 0.5)^{\Omega_2}(\eta + 0.5)^{\Omega_3}(\eta + 0.5)^{\Omega_4} \tag{19}
\]

with \( \alpha = u, v, w, \theta_u, \) and \( \theta_v \).

By applying the Ritz method, we minimize the total energy functional \( \Pi \) with respect to the unknown coefficients, \( \frac{\partial \Pi}{\partial c_i^u} = 0, \)

\( \frac{\partial \Pi}{\partial c_i^v} = 0, \)

\( \frac{\partial \Pi}{\partial c_i^w} = 0, \)

\( \frac{\partial \Pi}{\partial c_i^{\theta_u}} = 0, \)

\( \frac{\partial \Pi}{\partial c_i^{\theta_v}} = 0, \) \( \tag{20e} \)

for \( i \) from 1 to \( m \). The differentiation leads to an eigenvalue equation,

\[
\{K - \lambda M\} \{c\} = 0, \tag{21}
\]

where \( \{c\} = \{c^u, c^v, c^w, a e^{\theta_u}, b e^{\theta_v}\}^T \), and \( \lambda \) is the nondimensional eigenvalue defined as

\[
\lambda = \frac{\omega ab}{h} \sqrt{\frac{\lambda}{D_0}} \tag{22}
\]

with \( D_0 = Q_{11}/12 \). Details of the stiffness matrices \( K \) and the mass matrices \( M \) are shown in the Appendix.

**TABLE III.** Convergence study of frequency parameters, \( \lambda' = \omega h \sqrt{\rho/\mu G} \), for an isotropic thick plate with different degree of polynomial \( (p) \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \lambda' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.0930 0.2220 0.3407 0.4152 0.5256 0.6975 0.7510</td>
</tr>
<tr>
<td>9</td>
<td>0.0930 0.2220 0.3406 0.4151 0.5209 0.6846 0.7463</td>
</tr>
<tr>
<td>11</td>
<td>0.0930 0.2220 0.3406 0.4150 0.5208 0.6840 0.7454</td>
</tr>
<tr>
<td>13</td>
<td>0.0930 0.2220 0.3406 0.4151 0.5208 0.6840 0.7454</td>
</tr>
<tr>
<td>15</td>
<td>0.0930 0.2220 0.3406 0.4151 0.5208 0.6840 0.7454</td>
</tr>
<tr>
<td>CLPT</td>
<td>0.0963 0.2408 0.3853 0.4818 0.6261 0.8686 0.9632</td>
</tr>
<tr>
<td>Mindlin ( ^{16} )</td>
<td>0.0930 0.2218 0.3402 0.4144 0.5197 0.6821 0.7431</td>
</tr>
<tr>
<td>Malikarjuna ( ^{8,4} )</td>
<td>0.0929 0.2216 0.3379 0.4184 0.5152 0.6941 0.7610</td>
</tr>
<tr>
<td>Noor ( ^{17} )</td>
<td>0.0932 0.2226 0.3421 0.4171 0.5239 0.6889 0.7511</td>
</tr>
</tbody>
</table>

\( ^{4} \) Higher-order shear deformation theory.

**TABLE IV.** Convergence study of frequency parameters, \( \lambda' = \omega h \sqrt{\rho/\mu E} \), for a three-ply square laminate with different degree of polynomial \( (p) \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \lambda' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.4537 0.6883 0.7222 0.8449 1.0182 1.1807 1.2653</td>
</tr>
<tr>
<td>9</td>
<td>0.4538 0.6883 0.7220 0.8442 1.0173 1.1801 1.2634</td>
</tr>
<tr>
<td>11</td>
<td>0.4523 0.6883 0.7230 0.8438 1.0171 1.1799 1.2628</td>
</tr>
<tr>
<td>13</td>
<td>0.4520 0.6883 0.7230 0.8435 1.0170 1.1799 1.2624</td>
</tr>
<tr>
<td>15</td>
<td>0.4517 0.6883 0.7230 0.8434 1.0170 1.1799 1.2621</td>
</tr>
</tbody>
</table>
II. NUMERICAL STUDIES AND DISCUSSIONS

Several examples with various combinations of boundary conditions have been investigated to demonstrate the versatility of the \( p \)-Ritz method. All laminae have been assumed to have the same orthotropic properties and equal thicknesses. Numerical results from the published literature have been taken for comparison to demonstrate the accuracy of \( p \)-Ritz method integrated with the higher-order shear deformation theory. In this study, all results have been computed in double precision on a SGI PowerChallenge computer. Material properties used in all examples have been assumed to be nondimensional as follows:

Material 1: \( E_1/E_2 = 10.0 \), \( G_{12}/E_2 = (1 + \nu)/2 \), 
\( G_{23} = G_{13} = G_{12} \), \( \nu = 0.3 \).

Material 2: \( E_1/E_2 = 25.0 \), \( G_{12}/E_2 = 0.5 \), \( G_{23}/E_2 = 0.2 \), 
\( G_{13} = G_{12} \), \( \nu_{12} = 0.25 \).

Material 3: \( E_1/E_2 = 40.0 \), \( G_{12}/E_2 = 0.6 \), \( G_{23}/E_2 = 0.5 \), 
\( G_{13} = G_{12} \), \( \nu_{12} = 0.25 \).

Material 4: \( E_1/E_2 = 1.9040209 \), 
\( G_{12}/E_2 = 0.575868 \), 
\( G_{23}/E_2 = 0.658135 \).

\( G_{13} = 0.3391225 \), \( \nu_{12} = 0.44 \).

The present method has been verified using three examples with an aspect ratio \( a/b = 1.0 \) and boundary conditions of \( S_3 S_3 S_3 S_3 \). The first example is an isotropic square plate of material 1 and with a length-to-thickness ratio \( a/h = 10 \), while the second and the third examples are three-ply and five-ply laminated plates of material 3 with \( a/h = 5 \) and stacking sequence \((45°/−45°/45°)\) and \((45°/−45°/45°/−45°/45°)\), respectively. A convergence study for frequency parameters was carried out by increasing the number of polynomials from 7 to 15. In Table III, the results for an isotropic plate obtained using this method are very close to Noor’s 3-D solutions and Mallikarjuna’s solutions obtained by a higher-order theory and the finite element method. The frequency parameters obtained using \( p = 9 \) are within a discrepancy of 0.3% of \( p = 15 \) in all cases as shown in Tables III–V. It is observed that the fundamental frequencies for single-layer plates converge faster than those for the multi-layer laminates in Tables IV and V. The frequencies also converge rapidly for the modes dominated by in-plane displacements as opposed to the out-of-plane displacements. Higher values for \( p \) are therefore needed to provide sufficient accuracy for the higher modes. However, a higher \( p \) also leads to more computational effort. Therefore, the parameter \( p = 15 \) was adopted as a reasonable compromise for subsequent computation.

The fourth example presents a square plate made of ara-

TABLE V. Convergence study of frequency parameters, \( \lambda' = \omega h \sqrt{\rho E_2} \), for a five-ply square laminate with different degree of polynomial (\( p \)).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \lambda' )</th>
<th>( \lambda' )</th>
<th>( \lambda' )</th>
<th>( \lambda' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.4963</td>
<td>0.6883</td>
<td>0.8085</td>
<td>0.8778</td>
</tr>
<tr>
<td>9</td>
<td>0.4955</td>
<td>0.6883</td>
<td>0.8082</td>
<td>0.8772</td>
</tr>
<tr>
<td>11</td>
<td>0.4951</td>
<td>0.6883</td>
<td>0.8080</td>
<td>0.8769</td>
</tr>
<tr>
<td>13</td>
<td>0.4949</td>
<td>0.6883</td>
<td>0.8080</td>
<td>0.8767</td>
</tr>
<tr>
<td>15</td>
<td>0.4948</td>
<td>0.6883</td>
<td>0.8080</td>
<td>0.8765</td>
</tr>
</tbody>
</table>

TABLE VI. Comparison of frequency parameters, \( \lambda' = \omega h \sqrt{\rho E_2} \), for an orthotropic square plate made of Material 4.

<table>
<thead>
<tr>
<th>Source</th>
<th>( \lambda' )</th>
<th>( \lambda' )</th>
<th>( \lambda' )</th>
<th>( \lambda' )</th>
<th>( \lambda' )</th>
<th>( \lambda' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLPT</td>
<td>0.0493</td>
<td>0.1095</td>
<td>0.1327</td>
<td>0.1924</td>
<td>0.2070</td>
<td>0.2671</td>
</tr>
<tr>
<td>Mallikarjuna(^a)</td>
<td>0.0473</td>
<td>0.1032</td>
<td>0.1190</td>
<td>0.1662</td>
<td>0.1906</td>
<td>0.2205</td>
</tr>
<tr>
<td>Mallikarjuna(^b)</td>
<td>0.0474</td>
<td>0.1032</td>
<td>0.1190</td>
<td>0.1662</td>
<td>0.1906</td>
<td>0.2205</td>
</tr>
<tr>
<td>Present study</td>
<td>0.0474</td>
<td>0.1032</td>
<td>0.1188</td>
<td>0.1694</td>
<td>0.1888</td>
<td>0.2180</td>
</tr>
<tr>
<td>Reddy(^a)</td>
<td>0.0474</td>
<td>0.1032</td>
<td>0.1188</td>
<td>0.1694</td>
<td>0.1888</td>
<td>0.2180</td>
</tr>
<tr>
<td>Srinivas(^b)</td>
<td>0.0474</td>
<td>0.1032</td>
<td>0.1188</td>
<td>0.1694</td>
<td>0.1888</td>
<td>0.2180</td>
</tr>
</tbody>
</table>

\(^a\)First-order shear deformation theory.
\(^b\)Higher-order shear deformation theory.
gonite crystal with \( a/h = 10 \) and boundary conditions of \( S_3S_3S_3S_3 \). The mode shapes and frequency parameters are tabulated in Table VI and compared with available results. While the results obtained by the present method have an error of less than 0.25% with respect to both Sriniva’s 3-D solutions and Reddy’s closed form solutions, the errors in the results from classical laminate plate theory (CLPT), Mallikarjuna (first-order theory), and Mallikarjuna (higher-order theory) are 23.78%, 1.73%, and 1.01% of the exact solutions. As expected, the results obtained by using present method are superior to those of CLPT and first-order theories. They are also very close to the closed form solutions and 3-D elasticity solutions. Therefore, the \( p \)-Ritz method is able to provide very accurate results without the difficulties of mesh generations and discretization losses in the finite element method.

In the fifth example, a four-ply square laminate made of material 3 stacked in a sequence of \((0°/90°)\), with boundary condition \( S_2 \) at all edges and a length-to-thickness ratio of 10 has been examined. The effect of the length-to-thickness ratio on the laminated plates has been presented in Table VII by varying the ratio from 4 to 100. As shown in Table VII, the effect of length-to-thickness ratio on the fundamental frequencies is pronounced and the error in using the classical laminated plate theory increases for thicker plates because of the neglect of shear effects. The present results are in close agreement with Reddy’s closed form solutions.

The sixth example analyzed the same laminate as example 5 but with a length-to-thickness ratio of 5, different numbers of layers, and different degree of orthotropy. From the results in Table VIII, it is observed that the prediction of the fundamental frequency by CLPT is inaccurate for materials with a high degree of anisotropy. This reaffirms the fact that the effect of material anisotropy on the fundamental frequency for symmetrically laminated plates is pronounced. In addition, the response characteristics predicted by the present method are accurate and the maximum error in the fundamental frequencies is about 1% compared to the 3-D solutions of Noor.

The effect of boundary conditions on the fundamental frequency has been examined in the seventh example. A four-ply square laminate made of material 3 has been as-

### Table VII. Effect of length-to-thickness ratio \((a/h)\) on the fundamental frequency parameters, \(\lambda' = (\omega a^2/h)\sqrt{\rho E_2}\), for four-ply square plates of Material 3.

<table>
<thead>
<tr>
<th>Source</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLPT</td>
<td>17.907</td>
<td>18.215</td>
<td>18.652</td>
<td>18.767</td>
<td>18.799</td>
<td>18.804</td>
</tr>
<tr>
<td>Rikards(^7)</td>
<td>9.244</td>
<td>10.690</td>
<td>14.966</td>
<td>17.532</td>
<td>18.632</td>
<td>18.898</td>
</tr>
<tr>
<td>Mallikarjuna(^6,a)</td>
<td>9.227</td>
<td>10.736</td>
<td>15.073</td>
<td>17.628</td>
<td>18.672</td>
<td>18.835</td>
</tr>
<tr>
<td>Mallikarjuna(^6,b)</td>
<td>9.258</td>
<td>10.740</td>
<td>15.090</td>
<td>17.637</td>
<td>18.669</td>
<td>18.833</td>
</tr>
<tr>
<td>Present study</td>
<td>9.323</td>
<td>10.787</td>
<td>15.107</td>
<td>17.647</td>
<td>18.672</td>
<td>18.836</td>
</tr>
<tr>
<td>Reddy(^5)</td>
<td>9.369</td>
<td>10.820</td>
<td>15.083</td>
<td>17.583</td>
<td>18.590</td>
<td>18.751</td>
</tr>
</tbody>
</table>

\(^a\)First-order shear deformation theory.
\(^b\)Higher-order shear deformation theory.

### Table VIII. Effect of degree of orthotropy on the fundamental frequency, \(\lambda' = (\omega a^2/h)\sqrt{\rho E_2}\), for a square plates with \(a/h = 5\) and \(S_3S_3S_3S_2\) edge conditions.

<table>
<thead>
<tr>
<th>Source</th>
<th>3Layers</th>
<th>3</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLPT</td>
<td>4</td>
<td>0.2920</td>
<td>0.4126</td>
<td>0.5404</td>
<td>0.6434</td>
<td>0.7320</td>
</tr>
<tr>
<td>Reddy(^5)</td>
<td>5</td>
<td>0.2624</td>
<td>0.3309</td>
<td>0.3811</td>
<td>0.4109</td>
<td>0.4315</td>
</tr>
<tr>
<td>Present study</td>
<td></td>
<td>0.2624</td>
<td>0.3309</td>
<td>0.3811</td>
<td>0.4109</td>
<td>0.4315</td>
</tr>
<tr>
<td>Noor(^17)</td>
<td></td>
<td>0.2647</td>
<td>0.3284</td>
<td>0.3824</td>
<td>0.4109</td>
<td>0.4301</td>
</tr>
<tr>
<td>CPT</td>
<td>5</td>
<td>0.2920</td>
<td>0.4126</td>
<td>0.5404</td>
<td>0.6434</td>
<td>0.7320</td>
</tr>
<tr>
<td>Mallikarjuna(^6,a)</td>
<td>0.2626</td>
<td>0.3362</td>
<td>0.3919</td>
<td>0.4246</td>
<td>0.4463</td>
<td></td>
</tr>
<tr>
<td>Rikards(^7)</td>
<td>0.2608</td>
<td>0.3313</td>
<td>0.3852</td>
<td>0.4142</td>
<td>0.4340</td>
<td></td>
</tr>
<tr>
<td>Mallikarjuna(^6,b)</td>
<td>0.2626</td>
<td>0.3362</td>
<td>0.3919</td>
<td>0.4248</td>
<td>0.4470</td>
<td></td>
</tr>
<tr>
<td>Present study</td>
<td></td>
<td>0.2634</td>
<td>0.3372</td>
<td>0.3937</td>
<td>0.4274</td>
<td>0.4505</td>
</tr>
<tr>
<td>Noor(^17)</td>
<td></td>
<td>0.2659</td>
<td>0.3409</td>
<td>0.3979</td>
<td>0.4314</td>
<td>0.4537</td>
</tr>
<tr>
<td>CPT</td>
<td>9</td>
<td>0.2920</td>
<td>0.4126</td>
<td>0.5404</td>
<td>0.6434</td>
<td>0.7320</td>
</tr>
<tr>
<td>Mallikarjuna(^6,a)</td>
<td>0.2630</td>
<td>0.3404</td>
<td>0.4011</td>
<td>0.4376</td>
<td>0.4622</td>
<td></td>
</tr>
<tr>
<td>Mallikarjuna(^6,b)</td>
<td>0.2630</td>
<td>0.3404</td>
<td>0.4011</td>
<td>0.4376</td>
<td>0.4622</td>
<td></td>
</tr>
<tr>
<td>Present study</td>
<td></td>
<td>0.2638</td>
<td>0.3413</td>
<td>0.4024</td>
<td>0.4395</td>
<td>0.4648</td>
</tr>
<tr>
<td>Noor(^17)</td>
<td></td>
<td>0.2664</td>
<td>0.3443</td>
<td>0.4055</td>
<td>0.4421</td>
<td>0.4668</td>
</tr>
</tbody>
</table>

\(^a\)First-order shear deformation theory.
\(^b\)Higher-order shear deformation theory.
TABLE IX. Effect of boundary conditions on the fundamental frequency parameters, \( \lambda' = (\omega a / h) \sqrt{\rho E_z} \), for a four-ply square laminate of material 3.

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>Fundamental frequency parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1F_3C_2F_3 )</td>
<td>1.1378</td>
</tr>
<tr>
<td>( F_2F_3C_2F_3 )</td>
<td>1.1378</td>
</tr>
<tr>
<td>( S_2F_3S_2F_3 )</td>
<td>6.2185</td>
</tr>
<tr>
<td>( C_1F_3C_2F_3 )</td>
<td>12.9422</td>
</tr>
<tr>
<td>( S_2S_2S_2S_2 )</td>
<td>17.3813</td>
</tr>
<tr>
<td>( S_2S_2S_2C_2 )</td>
<td>17.6704</td>
</tr>
<tr>
<td>( S_2S_2S_2S_2 )</td>
<td>17.6704</td>
</tr>
<tr>
<td>( S_2S_2S_2C_2 )</td>
<td>17.9726</td>
</tr>
<tr>
<td>( S_3S_3C_3C_3 )</td>
<td>19.5025</td>
</tr>
<tr>
<td>( C_1C_1C_1C_1 )</td>
<td>21.6722</td>
</tr>
</tbody>
</table>

assumed to have a length-to-thickness ratio of 10 and be stacked with a sequence of \((-45^\circ/45^\circ)_6\). As summarized in Table IX, the frequencies are higher for plates with stiffer constraints.

In the last two examples, the combined effects of plate aspect ratio, length-to-thickness ratio, and stacking angle for symmetric angle-ply, cantilevered \((C_1F_4C_4F_4)_h\), and simply supported \((S_1S_1S_1S_1)_h\) laminated plate have been investigated. The laminates have been assumed to be made of material 3, and be either five-ply with stacking sequence \((\theta - \theta - \theta - \theta - \theta)\) or three-ply with stacking sequence \((\theta - \theta - \theta)\).

TABLE XI. Effect of plate aspect ratio \((a/b)\), length-to-thickness ratio \((a/h)\), and lamination angle \((\theta)\) on the fundamental frequency parameters, \(\lambda' = 100 \times (\omega a / h) \sqrt{\rho E_z}\), for laminated plates of material 3 with stacking sequence \((\theta - \theta - \theta - \theta - \theta)\) and \(C_1F_4F_4\) edge conditions.

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>( a/h )</th>
<th>( \theta )</th>
<th>Material 2</th>
<th>Material 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5</td>
<td>1.0</td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td>5</td>
<td>32.2071</td>
<td>35.8155</td>
<td>42.6236</td>
<td>52.0862</td>
</tr>
<tr>
<td>30</td>
<td>29.0077</td>
<td>36.2748</td>
<td>46.6866</td>
<td>58.8802</td>
</tr>
<tr>
<td>45</td>
<td>25.6464</td>
<td>36.6625</td>
<td>50.3993</td>
<td>66.1334</td>
</tr>
<tr>
<td>60</td>
<td>22.1526</td>
<td>36.2748</td>
<td>53.6901</td>
<td>72.9672</td>
</tr>
<tr>
<td>75</td>
<td>18.5448</td>
<td>38.8515</td>
<td>56.3615</td>
<td>78.2015</td>
</tr>
<tr>
<td></td>
<td>11.2081</td>
<td>12.4252</td>
<td>14.9253</td>
<td>18.5448</td>
</tr>
<tr>
<td>30</td>
<td>9.8732</td>
<td>12.6716</td>
<td>16.9729</td>
<td>22.1526</td>
</tr>
<tr>
<td>45</td>
<td>8.3602</td>
<td>12.8639</td>
<td>18.7282</td>
<td>25.6464</td>
</tr>
<tr>
<td>60</td>
<td>6.9411</td>
<td>12.6716</td>
<td>20.3417</td>
<td>29.2070</td>
</tr>
<tr>
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<td>5.5211</td>
<td>12.4252</td>
<td>21.8693</td>
<td>32.2071</td>
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<td>11.2081</td>
<td>12.4252</td>
<td>14.9253</td>
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<tr>
<td>30</td>
<td>9.8732</td>
<td>12.6716</td>
<td>16.9729</td>
<td>22.1526</td>
</tr>
<tr>
<td>45</td>
<td>8.3602</td>
<td>12.8639</td>
<td>18.7282</td>
<td>25.6464</td>
</tr>
<tr>
<td>60</td>
<td>6.9411</td>
<td>12.6716</td>
<td>20.3417</td>
<td>29.2070</td>
</tr>
<tr>
<td>75</td>
<td>5.5211</td>
<td>12.4252</td>
<td>21.8693</td>
<td>32.2071</td>
</tr>
<tr>
<td>20</td>
<td>11.2081</td>
<td>12.4252</td>
<td>14.9253</td>
<td>18.5448</td>
</tr>
<tr>
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<td>9.8732</td>
<td>12.6716</td>
<td>16.9729</td>
<td>22.1526</td>
</tr>
<tr>
<td>45</td>
<td>8.3602</td>
<td>12.8639</td>
<td>18.7282</td>
<td>25.6464</td>
</tr>
<tr>
<td>60</td>
<td>6.9411</td>
<td>12.6716</td>
<td>20.3417</td>
<td>29.2070</td>
</tr>
<tr>
<td>75</td>
<td>5.5211</td>
<td>12.4252</td>
<td>21.8693</td>
<td>32.2071</td>
</tr>
<tr>
<td>30</td>
<td>11.2081</td>
<td>12.4252</td>
<td>14.9253</td>
<td>18.5448</td>
</tr>
<tr>
<td>30</td>
<td>9.8732</td>
<td>12.6716</td>
<td>16.9729</td>
<td>22.1526</td>
</tr>
<tr>
<td>45</td>
<td>8.3602</td>
<td>12.8639</td>
<td>18.7282</td>
<td>25.6464</td>
</tr>
<tr>
<td>60</td>
<td>6.9411</td>
<td>12.6716</td>
<td>20.3417</td>
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<tr>
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<td>12.6716</td>
<td>16.9729</td>
<td>22.1526</td>
</tr>
<tr>
<td>45</td>
<td>8.3602</td>
<td>12.8639</td>
<td>18.7282</td>
<td>25.6464</td>
</tr>
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<td>60</td>
<td>6.9411</td>
<td>12.6716</td>
<td>20.3417</td>
<td>29.2070</td>
</tr>
<tr>
<td>75</td>
<td>5.5211</td>
<td>12.4252</td>
<td>21.8693</td>
<td>32.2071</td>
</tr>
<tr>
<td>50</td>
<td>11.2081</td>
<td>12.4252</td>
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</tr>
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<tr>
<td>75</td>
<td>5.5211</td>
<td>12.4252</td>
<td>21.8693</td>
<td>32.2071</td>
</tr>
</tbody>
</table>
results are presented in Tables X and XI. In both examples, the fundamental frequencies decrease with an increase in length-to-thickness ratio. It is observed that boundary conditions have a significant influence on the effect of stacking angle for laminated plates. As shown in Table XI, for the simply supported laminates, the fundamental frequencies decrease as stacking angle increases for a plate with $a/b < 1.0$, and increase for $a/b > 1.0$. However, for cantilevered laminates, the fundamental frequencies decrease as the stacking angle increases. It has also been shown that the fundamental frequencies increase with aspect ratio for simply supported laminates and decrease with aspect ratio for cantilevered laminates.

III. CONCLUSIONS

The $p$-Ritz method has been employed for free vibration analysis of thick composite plates with symmetric lamination based on the higher-order shear deformation theory. A concise governing eigenvalue equation has been derived. Convergence of eigenvalues has been verified and excellent agreement has been achieved with respect to first-order, higher-order, finite element, and three-dimensional elasticity solutions. Numerical frequencies for laminates made of materials with different degrees of orthotropy have been presented and illustrated with relevant vibration mode shapes. The effect of length-to-thickness ratio, boundary conditions, plate aspect ratio, number of layers, and stacking angles on the laminates have also been investigated. This analysis suggests that, so far as the free vibration of laminated composite plate is concerned, the higher-order shear deformation theory is able to predict accurate solutions. The versatility of the $p$-Ritz method in accounting for laminated plates with a variety of boundary constraints should be appreciated.

APPENDIX

Substitution of Eqs. (1a)–(1c), (2), (4), and (10) into Eqs. (8) and (9) yields

\[
U = \frac{1}{2} \int_{A} \left[ A_{55} \Phi_{s}^2 + D_{11} \left( \frac{\partial \Phi_{s}}{\partial x} \right)^2 + 2D_{16} \left( \frac{\partial \Phi_{s}}{\partial y} \right)^2 + D_{66} \left( \frac{\partial \Phi_{s}}{\partial y} \right)^2 + 2A_{45} \Phi_{s} \Phi_{y} + A_{44} \Phi_{y}^2 + 2D_{16} \left( \frac{\partial \Phi_{s}}{\partial x} \right) \left( \frac{\partial \Phi_{s}}{\partial y} \right) \right] + \frac{1}{2} \int_{A} \left[ F_{44} \left( \frac{\partial \Phi_{s}}{\partial y} \right) \left( \frac{\partial \Phi_{s}}{\partial x} \right) + F_{45} \left( \frac{\partial \Phi_{s}}{\partial x} \right) \left( \frac{\partial \Phi_{s}}{\partial y} \right) - D_{55} \left( \frac{\partial \Phi_{s}}{\partial x} \right)^2 + 2F_{16} \left( \frac{\partial \Phi_{s}}{\partial y} \right)^2 + D_{26} \left( \frac{\partial \Phi_{s}}{\partial x} \right)^2 + D_{26} \left( \frac{\partial \Phi_{s}}{\partial y} \right)^2 + D_{26}^2 \left( \frac{\partial \Phi_{s}}{\partial x} \right) \left( \frac{\partial \Phi_{s}}{\partial y} \right) \right]
\]
The stiffness matrices $K$ and the mass matrices $M$ are given by

$$K = \frac{1}{D_0} \begin{bmatrix} [K_{uu}] & [K_{uw}] & 0 & 0 & 0 \\ [K_{uw}] & [K_{ww}] & 0 & 0 & 0 \\ \text{sym.} & [K_{w,\theta_1}] & [K_{w,\theta_2}] \\ & [K_{w,\theta_1}] & [K_{w,\theta_2}] \\ & [K_{w,\theta_2}] & [K_{w,\theta_3}] \end{bmatrix}$$

and

$$M = \begin{bmatrix} [M_{uu}] & 0 & 0 & 0 & 0 \\ [M_{uw}] & 0 & 0 & 0 & 0 \\ \text{sym.} & [M_{w,\theta_1}] & [M_{w,\theta_2}] & [M_{w,\theta_3}] \\ & [M_{w,\theta_2}] & [M_{w,\theta_3}] \\ & [M_{w,\theta_3}] \end{bmatrix},$$

where the elements of $K$ and $M$ can further be expressed as

$$K_{uu} = A_{66} \left( \frac{a^2}{h^3} \right) R_{e_i e_j}^{0101} + A_{16} \left( \frac{b^2}{h^3} \right) R_{e_i e_j}^{0110} + A_{11} \left( \frac{b^2}{h^3} \right) R_{e_i e_j}^{1010},$$  

$$K_{uw} = A_{26} \left( \frac{a^2}{h^3} \right) R_{e_i e_j}^{0101} + A_{26} \left( \frac{b^2}{h^3} \right) R_{e_i e_j}^{0110} + A_{12} \left( \frac{b^2}{h^3} \right) R_{e_i e_j}^{1010},$$  

$$K_{w,\theta} = A_{26} \left( \frac{a^2}{h^3} \right) R_{e_i e_j}^{0101} + A_{26} \left( \frac{b^2}{h^3} \right) R_{e_i e_j}^{0110} + A_{12} \left( \frac{b^2}{h^3} \right) R_{e_i e_j}^{1010}.$$
\[ K_{ij}^{ww} = \left[ A_{44} \left( \frac{a^2}{h^2} \right) - D_{44} \left( \frac{8a^2}{h^3} \right) + F_{44} \left( \frac{16a^2}{h^4} \right) \right] R_{0101}^{0101} e_i^p e_j^p + \left[ A_{45} \left( \frac{ab}{h^2} \right) - D_{45} \left( \frac{8ab}{h^3} \right) + F_{45} \left( \frac{16ab}{h^4} \right) \right] \left[ R_{0101}^{0101} e_i^p e_j^p + \left[ A_{55} \left( \frac{b^2}{h^2} \right) - D_{55} \left( \frac{8b^2}{h^3} \right) + F_{55} \left( \frac{16b^2}{h^4} \right) \right] \right] + H_{26} \left( \frac{32a}{9bh} \right) \left[ R_{0202}^{0201} e_i^p e_j^p + H_{20} \left( \frac{32b}{9ah} \right) \right] \left[ R_{0110}^{0110} e_i^p e_j^p \right] + H_{16} \left( \frac{16b}{9a^2h} \right) \left[ R_{1120}^{1120} e_i^p e_j^p \right] + H_{12} \left( \frac{16}{9h^2} \right) \left[ R_{1111}^{1111} e_i^p e_j^p \right], \quad (A5d) \]

\[ K_{ij}^{ww} = \left[ A_{45} \left( \frac{ab}{h^2} \right) - D_{45} \left( \frac{8ab}{h^3} \right) + F_{45} \left( \frac{16ab}{h^4} \right) \right] R_{0100}^{0100} e_i^p e_j^p + \left[ H_{26} \left( \frac{16a}{9bh} \right) - F_{26} \left( \frac{4a}{3h^3} \right) \right] + \left[ H_{12} \left( \frac{16}{9h^2} \right) \right] \left[ R_{0101}^{0101} e_i^p e_j^p \right] + H_{16} \left( \frac{16b}{9a^2h} \right) \left[ R_{1110}^{1110} e_i^p e_j^p \right], \quad (A5e) \]

\[ K_{ij}^{ww} = \left[ A_{44} \left( \frac{a^2}{h^2} \right) - D_{44} \left( \frac{8a^2}{h^3} \right) + F_{44} \left( \frac{16a^2}{h^4} \right) \right] R_{0000}^{0000} e_i^p e_j^p + \left[ H_{66} \left( \frac{32}{9h^3} \right) \right] R_{0100}^{0100} e_i^p e_j^p + \left[ H_{12} \left( \frac{16}{9h^2} \right) \right] \left[ R_{0101}^{0101} e_i^p e_j^p \right] + \left[ D_{26} \left( \frac{8a}{3b^3} \right) \right] \left[ R_{0100}^{0100} e_i^p e_j^p \right], \quad (A5f) \]

\[ K_{ij}^{ww} = \left[ A_{55} \left( \frac{b^2}{h^2} \right) - D_{55} \left( \frac{8b^2}{h^3} \right) + F_{55} \left( \frac{16b^2}{h^4} \right) \right] R_{0000}^{0000} e_i^p e_j^p + \left[ H_{66} \left( \frac{32}{9h^3} \right) \right] R_{0100}^{0100} e_i^p e_j^p + \left[ H_{12} \left( \frac{8}{3h^3} \right) \right] \left[ R_{0101}^{0101} e_i^p e_j^p \right] + \left[ D_{16} \left( \frac{b}{ah} \right) \right], \quad (A5g) \]

\[ K_{ij}^{ww} = \left[ A_{45} \left( \frac{ab}{h^2} \right) - D_{45} \left( \frac{8ab}{h^3} \right) + F_{45} \left( \frac{16ab}{h^4} \right) \right] R_{0000}^{0000} e_i^p e_j^p + \left[ D_{26} \left( \frac{8a}{3b^3} \right) \right] \left[ R_{0101}^{0101} e_i^p e_j^p \right] + H_{16} \left( \frac{16b}{9a^2h} \right) \left[ R_{0110}^{0110} e_i^p e_j^p \right], \quad (A5h) \]

\[ K_{ij}^{ww} = \left[ A_{44} \left( \frac{a^2}{h^2} \right) - D_{44} \left( \frac{8a^2}{h^3} \right) + F_{44} \left( \frac{16a^2}{h^4} \right) \right] R_{0000}^{0000} e_i^p e_j^p + \left[ H_{22} \left( \frac{16a}{9bh} \right) - F_{22} \left( \frac{4a}{3h^3} \right) \right] \left[ R_{0110}^{0110} e_i^p e_j^p \right] + \left[ D_{16} \left( \frac{b}{ah} \right) \right], \quad (A5i) \]

and

\[ M_{ij}^{ww} = h R_{0000}^{0000} e_i^p e_j^p, \quad (A6a) \]

\[ M_{ij}^{ww} = h R_{0000}^{0000} e_i^p e_j^p, \quad (A6b) \]

\[ M_{ij}^{ww} = h R_{0000}^{0000} e_i^p e_j^p, \quad (A6c) \]

\[ M_{ij}^{ww} = h R_{0000}^{0000} e_i^p e_j^p, \quad (A6d) \]

\[ M_{ij}^{ww} = h R_{0000}^{0000} e_i^p e_j^p, \quad (A6e) \]

\[ M_{ij}^{ww} = h R_{0000}^{0000} e_i^p e_j^p, \quad (A6f) \]

\[ M_{ij}^{ww} = h R_{0000}^{0000} e_i^p e_j^p, \quad (A6g) \]
in which

\[
R_{eff}^{i\alpha j\beta} = \int \int_A \frac{\partial^{s+e} \psi_i^{\alpha} (\xi, \eta) \partial^{s+e} \psi_j^{\beta} (\xi, \eta)}{\partial \xi^s \partial \eta^e \partial \xi^s \partial \eta^e} \, d\xi \, d\eta.
\]  

(A7)


