Dynamic Traffic Assignment: Model Classifications and Recent Advances in Travel Choice Principles

W.Y. SZETO [a], S.C. WONG [a,b]

[a] Department of Civil Engineering, The University of Hong Kong, Hong Kong, China
[b] E-mail: hhecwsc@hkucc.hku.hk

Abstract: Dynamic Traffic Assignment (DTA) has been studied for more than four decades and numerous reviews of this research area have been conducted. This review focuses on the travel choice principle and the model classifications of DTA, and is supplementary to the existing reviews. The implications of the travel choice principle for the existence and uniqueness of solutions of DTA are discussed, and the interrelation between the travel choice principle and the traffic flow component is explained using the nonlinear complementarity problem, the variational inequality problem, the mathematical programming problem, and the fixed point problem formulations. This paper also points out that all of the existing travel choice principles reviewed are extended from those used in static traffic assignment and that there are many classifications of DTA models, in which each classification addresses one aspect of DTA modeling. Finally, some future research directions are identified.

Keywords: Dynamic traffic assignment, user equilibrium, route choice, departure time choice

1. Introduction

Dynamic Traffic Assignment (DTA) is a generalization of static traffic assignment. In simple terms, static traffic assignment is a problem of determining the number of vehicles entering each highway in a specific area per hour (i.e. the vehicular traffic flow or flow pattern on each highway), given the vehicular demand for travel from each of the origins to each of the destinations in the area. In other words, the problem is to assign traffic to different highways according to certain behavioral rules. However, this problem cannot capture the realistic changes in the number of vehicles on the highways over time or the departure time choices of travelers. Therefore, DTA generalizes static traffic assignment to determine the time-varying flow on each highway over a study period, given the overall demand for vehicular travel.

A simple example of DTA is as follows. Figure 1 depicts a road network with two nodes and two links. Node A represents the origin and node B represents the destination. The links represent the highways connecting the origin and the destination. Any driver can go from A to B by car via one of the two routes, i.e., via either Link 1 or Link 2. However, the minimum travel time via Link 2 is 30 minutes less than that via Link 1. If the arrival rate of vehicles at the bottleneck in the middle of Link 2 at any instant is not greater than the capacity (i.e., the maximum number of vehicles that can pass through the bottleneck per hour) of 2000 vehicles per hour, all vehicles can pass through the bottleneck without delay and their travel time is 30 minutes. Otherwise, a queue is formed behind the bottleneck and the travel time via Link 2 is increased. The longer the queue, the higher the travel time. There is also a bottleneck in Link 1 with a capacity of 4000 vehicles per hour and the minimum travel time via Link 1 is 1 hour. It is known that between 6:00 am and 10:00 am, a total of 8000 drivers travel to B from A along both routes. All of these drivers must reach B on or before 9:00 am. These drivers have a choice of departure time in addition to link (or route). They can select a departure time so that they arrive at B at 9:00 pm sharp but they waste a lot of time in queuing. They can also depart early to have less queuing time (or waiting time in queue) and arrive at B early. However, arriving at B too early is not desirable as the time between the arrival time and 9:00 am is wasted at B, because the time can be reserved for other activities. Given that the demand from A to B during that period is 8000 vehicles, the problem is to determine the numbers of vehicles using Links 1 and 2 over the study period. In other words, the problem is to find out the time-varying demand splits. Note that the splits depend on the travel times on both links and the travel times also depend on the splits.

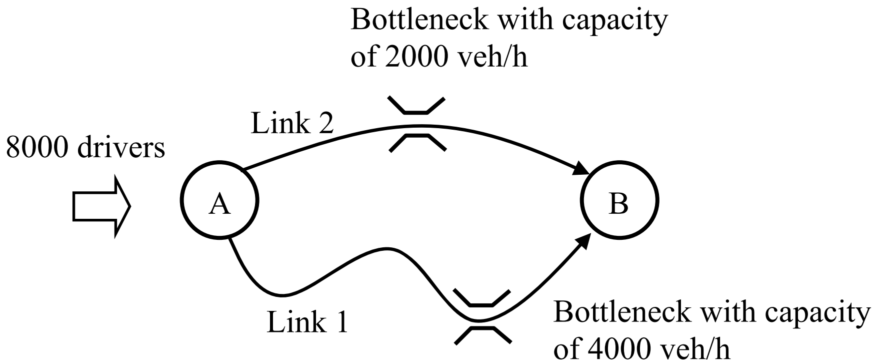

Figure 1 Example network

Figure 2 represents a solution for this simple example. The cumulative arrival curves represent the total number of drivers entering the links over time, whereas the cumulative departure curves represent the total number of drivers leaving the links over time. The vertical distance between the cumulative arrival and departure curves at a particular time gives the number of vehicles on the link at that time. The horizontal distance between two curves gives the travel time of a particular driver. As the minimum travel time on Link 2 is initially less than that on Link 1, drivers initially select Link 2. As the arrival rate of Link 2 is greater than the capacity of the bottleneck, the travel time on this link increases until it is equal to the minimum travel time of Link 1 of 60 minutes. Then, both links are chosen by drivers and all drivers can reach B before 9:00 am. As can be seen in Figure 2, the vertical distance between the cumulative arrival and departure curves is changing over time, meaning that the numbers of vehicles on the two links are changing over time. This is because the queuing time changes over time, which affects the departure time and route choice of drivers. The end result is that the minimum travel time from A to B is 30 minutes and the maximum travel time is 90 minutes. Some drivers depart earlier to have less travel time and queuing time. The first driver to leave A can travel to B without facing congestion and arrives at B at 7:00 am, whereas the last driver leaves A at 7:30 am and requires a travel time of 90 minutes to arrive at B sharply at 9:00 am. Furthermore, the number of drivers using each of the two links (which is the height of the curve) is equal to 4000 and the sum is equal to the demand of 8000 vehicles. However, in general, the usage of each link may not be the same.

This example also illustrates that DTA consists of two main components, namely *travel choice* and *traffic flow*. The travel choice component determines the traffic flow level on each road at each instant of time, given the road network performance in terms of the time-varying travel times on each road. The traffic flow component depicts how vehicular traffic propagates inside a road network, given the demand split to each route over time, and governs the performance of the road network, in the sense that more traffic on a link results in a higher travel time. The output of the travel choice component is the input of the traffic flow component while the output of the traffic flow component is the input of the travel choice component. DTA is then used to determine the flow pattern that satisfies the two components simultaneously.

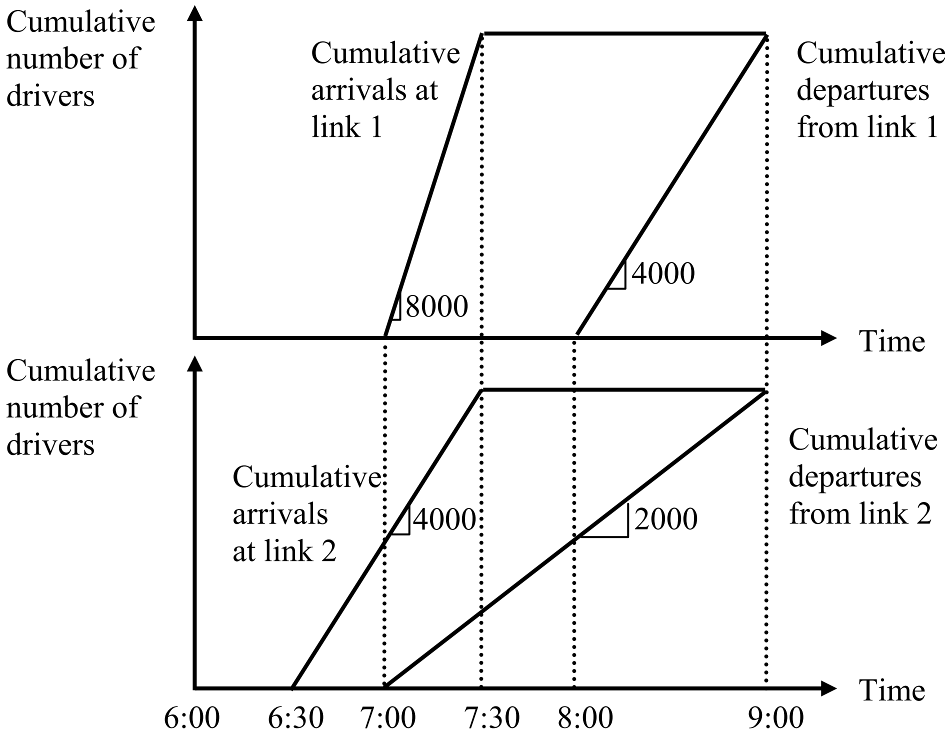

Figure 2 Cumulative arrivals and departures on Links 1 and 2

In this example, the demand of 8000 vehicles is required to propagate on either one of the two links. The travel choice component determines the split of the demand (or the number of vehicles entering each link) over time, based on the time-varying travel time of each link. Then, the traffic flow component propagates each vehicle on the respective links and determines the travel time. The travel time on each link over time must be the same as that used to determine the time-varying demand splits if the splits are optimal. Otherwise, a new set of splits based on the output of the traffic flow component should be used to determine the traffic flow on each link over time.

While the above example is simple, DTA is generally a difficult problem, especially when the networks are large and the study periods are long. This is because the number of routes can be huge, even for a medium size network, and the route set chosen by drivers can be time-variant. When actual traffic behavior, such as queues spilling backward and lane changing, is captured, the problem becomes even more difficult.

Although DTA is a difficult problem, it is practically important because DTA models have a wide range of applications. DTA models can be used for offline transportation planning and policy evaluation, and real-time traffic management, such as:

- managing the congestion of freeways through ramp metering [1, 2],
- controlling signal light setting [3, 4],
- advising routes for travelers equipped with global positioning system or advanced traveler information systems [e.g., 5, 6],

- determining time-varying toll levels and the charging locations [e.g., 7, 8],
- determining whether a new highway should be added or whether an existing highway should be expanded [e.g., 9],
- forecasting the usage of highways for future scenarios [e.g., 10], and
- evaluating the benefits of congestion mitigation schemes [e.g., 11].

In these applications, DTA models are used to predict the dynamic traffic flow pattern in a study area, given a demand scenario with a transportation planning or traffic management strategy. Very often, this flow pattern is an important input to another model for determining the best planning or management strategy. In other words, DTA models are core components of transportation planning and traffic management models.

Probably because of its wide application and complexity, the problem of DTA has drawn much attention in the literature. To illustrate this point, we conducted a keyword search of the SCOPUS database on 10 May 2011 to determine the number of DTA-related publications published each year. Other than the keyword *dynamic traffic assignment*, the final keywords chosen are related to the five categories namely, travel choice, traffic flow component, traffic simulation and software, DTA applications, and the queue modeling approach, as summarized in Table 1. Note that the traffic flow simulation software can be regarded as a traffic flow component of DTA. However, using those keywords alone will result in including non-transport papers in the search result. Therefore, we excluded those papers with keywords for non-transport networks, logistics and telecommunication. These keywords are listed below:

- Non-transport networks: *neural*, *logistic*, *photonic*, *social*, *optical*, *communication*, *radio*, *electric*, *cellular*, *W-ATM MAC*, *LAN*, *IT*, *information*, and *ATM networks*, and
- Logistics and telecommunication: *broadband*, *vehicle routing and scheduling*, *vehicle routing*, *vehicle scheduling*, *freight transport*, *protocol*, and *communication subsystem*.

Then, we read the titles and abstracts of the papers obtained to ensure that the papers are either addressing DTA or applying DTA models. A total of 1471 DTA-related publications were eventually found, of which 859 are journal articles. The earliest article was found to be published in 1971 by Yagar [12].

Table 1 Keywords used for literature review

Category	**Keywords**
Travel choice	dynamic user equilibrium, dynamic system optimal, dynamic system optimum, simultaneous route and departure time choice, simultaneous departure time and route choice
Traffic flow component	cell transmission model, link transmission model, traffic flow model, whole link model, dynamic link performance function, Vickrey's model, Vickrey's bottleneck model
Traffic simulation models & software	dynamic traffic simulation model, TRANSIMS, PARAMICS, VISSIM, DYNASMART, DynaMIT, CONTRAM, MITSIM
DTA applications	dynamic network design, dynamic equilibrium network design, dynamic pricing, dynamic toll, dynamic signal control, dynamic signal setting, dynamic OD demand estimation
The queue modeling approach	point queue, physical queue

Figure 3 shows the numbers of DTA-related publications for each year from 1971-2011. During 1971-1992, the numbers of DTA-related publications remains roughly constant at below 10 per year. From 1993 onwards, however, the numbers of DTA related publications increase almost monotonically. In 2009, for instance, there were over 200 publications, in contrast to about 20 in 1993. In 2010, the number was more than 180. These results seem to indicate that the number of DTA related publications is still increasing.

Table 2 shows the numbers and percentages of papers appearing in the top 3 journals and proceedings. According to this table, Transportation Research Record ranks number 1, with a share of 21% of all DTA related journal papers, whereas the IEEE Conference on Intelligent Transportation Systems Proceedings ranks number 1, with a share of 12% of all DTA-related proceedings papers. Good journals such as Transportation Science and Transportation Research Part C are not in the top three and only have a share of 2-3%. Most of the papers in the Proceedings of International Symposium on Transportation and Traffic Theory are not included in SCOPUS and hence we exclude those papers in Table 2.

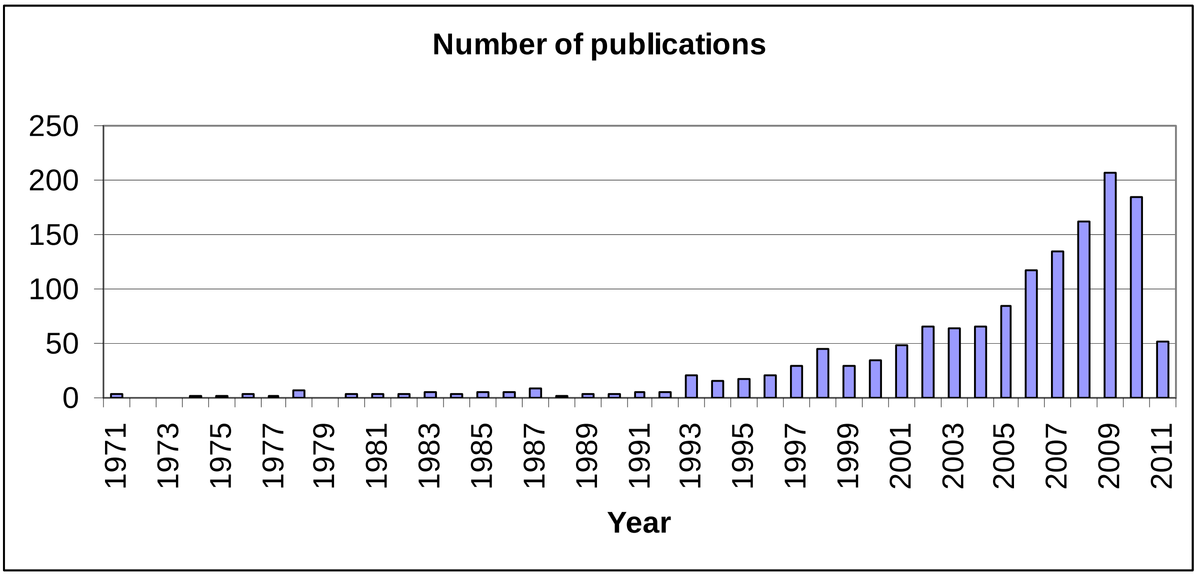

Figure 3 The number of DTA-related publications over time

Table 2 Top three journals and proceedings with respect to DTA related papers

Table 2a Journals

Rank	Journal	No. of papers	Overall percentage
1	Transportation Research Record	179	21%
2	Transportation Research Part B- Methodological	69	8%
3	Physica A: Statistical Mechanics and its Applications	33	4%

Table 2b Proceedings

Rank	Proceedings	No. of papers	Overall percentage
1	IEEE Conference on Intelligent Transportation Systems Proceedings (ITSC)	75	12%
2	Proceedings of the Conference on Traffic and Transportation Studies (ICTTS)	27	4%
3	Proceedings of the International Conference on Applications of Advanced Technologies in Transportation Engineering	20	3%

Given the huge number of DTA-related papers, various review studies have been conducted to summarize the literature and to give future research directions. To the best of our knowledge, at least eight relatively comprehensive DTA reviews have been conducted thus far. Table 3 shows the reviews and their focuses. However, these reviews have not focused on the travel choice principle adopted in DTA models, which is one of the important components of DTA models for obtaining realistic solutions to practical applications. Ideally, this travel choice component should be behaviorally sound and reflect the route/departure time choice behavior of travelers. Moreover, the classifications of DTA have not been summarized. We believe that it is important to review different classifications to highlight the various considerations in DTA modeling, including realistic representation, solution efficiency and ease of analyzing the problem.

Table 3 Recent comprehensive DTA reviews and their focuses

	Review	**Focus**
1	Cascetta and Cantarella [13]	focused on the DTA models and solution methods developed before 1991
2	Peeta and Ziliaskopoulos [14]	examined DTA papers published before 2000
3	Boyce et al. [15]	addressed analytical DTA formulations, with a focus on the variational inequality approach
4	Szeto and Lo [16]	compared the properties of DTA with different forms of traffic flow models, discussed their implications, and suggested future research directions
5	Szeto and Lo [17]	addressed the properties of DTA problems with and without considering the effects of spatial queues, and discussed their implications
6	Mun [18]	addressed the traffic flow component of DTA
7	Jeihani [19]	focused on the DTA models used in some well-known computer packages such as TRANSIMS, PARAMICS, VISSIM, DYNASMART, DynaMIT, and CONTRAM
8	Szeto [20]	outlined the latest developments in one type of DTA models, namely cell-based dynamic equilibrium models, of which the first model was proposed in 1999

This paper reviews recent advances in the principles of travel choice and outlines the various DTA classifications, and is intended to serve as a supplementary reference to the existing reviews. In addition, the paper discusses the implications of the travel choice principles for the solution existence and uniqueness of the DTA model, and explains how the travel choice and traffic flow components are integrated. Some trends are identified and the limitations of existing models are briefly mentioned.

The remainder of this paper proceeds as follows. Section 2 reviews the travel choice component used in static traffic assignment and DTA. Section 3 discusses and classifies various DTA models. Finally, Section 4 gives concluding remarks and potential future research directions.

2. Travel choice component

Till now, the travel choice component of DTA is developed based on the route choice principle of static traffic assignment. Table 4 summaries the travel choice principles used in the travel choice components of static traffic assignment and DTA. The assumptions and criteria used to define the principles can be seen to vary.

2.1 Wardrop's principles for static traffic assignment

Traditionally, the travel choice component of DTA is developed based on Wardrop's first and second principles of static traffic assignment [21] (see also Type 1 in Table 4). *Wardrop's first principle* or *the user equilibrium (UE) principle* states that *the journey times on all routes actually used are equal and are not greater than those which would be experienced by a single vehicle on any unused route.* In other words, the travel times of all used routes between the same origin-destination (OD) pair are equal and minimal. This principle assumes that each traveler is identical, non-cooperative and rational in selecting the shortest route, and knows the exact travel time he/she will encounter. If all travelers select routes according to this principle the road network will be at equilibrium, such that no one can reduce their travel times by unilaterally choosing another route of the same OD pair. This principle has been extended to consider generalized travel cost instead of travel time, where generalized travel cost can include the monetary cost of in-vehicle travel time, tolls, parking charges, and fuel consumption costs, etc.

Table 4 Summary of equilibrium principles

Type	**Equilibrium Principle**	**Criterion in defining equilibrium**	**Perception error extension**	**Uncertain travel time extension**	**Dynamic extension**	**Bounded rationality extension**
1	User equilibrium (UE) [21]	Travel time				
	System optimal (SO) [21]	Marginal travel time				
2	Stochastic user equilibrium (SUE) [23]	Perceived travel time	√			
3	Risk user equilibrium [24, 25]	Effective travel time, travel time budget		√		
	Risk system optimum [26]	Marginal travel time budget				
	Mean excess traffic equilibrium (METE) [29]	Mean excess travel time (METT)				
	Reliability-based user equilibrium (RBUE) [32]	Normalized path preference index				
	Percentile equilibrium [33]	Percentile travel time				
	Risk-averse user equilibrium [35]	Expected travel time				
	Robust user equilibrium [40]	Worst case travel time				
	Prospect-based user equilibrium [47]	Travel prospect value				
4	Generalized traffic equilibrium [48]	Perceived expected disutility	√	√		
	Reliability-based	Perceived travel				

	stochastic user equilibrium (RSUE) [49]	time budget				
	Stochastic METE [50]	Perceived METT				
5	Dynamic user equilibrium (DUE) route choice [55]	Travel time	√			
	DUE departure time choice [56]	Generalized travel cost				
	DUE route/departure time choice [57]	Generalized travel cost				
	Dynamic system optimal (DSO) [58]	Marginal travel time				
6	Stochastic dynamic user equilibrium (SDUE) departure choice [59]	Perceived travel cost	√		√	
	SDUE route choice [60]	Perceived travel time				
	SDUE route/departure time choice [61]	Perceived generalized travel cost				
7	Dynamic generalized traffic equilibrium [62]	Perceived expected disutility	√	√	√	
	Reliability-based stochastic dynamic user equilibrium (RSDUE) route choice [63]	Perceived effective travel time				
8	Boundedly rational user equilibrium [67]	Travel time & travel time difference threshold				√

Mathematically, the UE conditions can be expressed as the following complementarity conditions:

$$f_p^{rs} \geq 0, \forall p, rs \text{ ,} \tag{1}$$

$$n_p^{rs} - u^{rs} \geq 0, \forall p, rs \text{ ,} \tag{2}$$

$$f_p^{rs}\left(n_p^{rs} - u^{rs}\right) = 0, \forall p, rs \text{ ,} \tag{3}$$

where f_p^{rs} is the flow on route or path p between OD pair *rs*; n_p^{rs} represents the travel time (or generalized travel cost) of route *p* between OD pair *rs*; and u^{rs} is the minimum travel time (or generalized cost) between OD pair *rs*. Condition (1) is the flow non-negativity constraint, which means that the flow must be nonnegative. Condition (2) defines the minimum travel time for an OD pair, which must not be greater than the travel time of each route connecting that OD pair. Condition (3) requires that if route *p* carries flow (i.e. $f_p^{rs} > 0$), the travel time on this route must be equal to the minimum travel time.

UE can also be explained from the economic concept of utility maximization [22], where utility measures the degree of satisfaction travelers derive from their choices. In the simplest case, a traveler's utility equals his/her budget or income minus the travel time. In this sense, the assumption of Wardrop's first principle can be viewed as travelers selecting routes to maximize their individual utility and, at equilibrium, no traveler can change his/her route to obtain a higher utility.

Wardrop's second principle or *the system optimal (SO) principle* states that *at equilibrium the* *total journey time is minimized* [21]. This means that each traveler behaves cooperatively in choosing his/her route to ensure that the total travel time of all travelers is minimized. This principle is useful in planning large traffic studies, where traffic management techniques, such as signal timing, lane allocations, and road pricing, are used to discourage or encourage traffic so that the total travel time is at a minimum. This principle is also useful for depicting logistic flow as the route choice of the flow is centrally controlled by a logistic system manager. The SO principle can also be formulated as a complementarity condition using marginal route travel time, which is the derivative of total travel time on a route with respect to route flow.

2.2 Stochastic extensions of Wardrop's principles for static traffic assignment

To capture more realistic travel behavior, Wardrop's first principle has been extended or improved in two stochastic dimensions:

1. travelers' perceptions of travel time/cost (see Type 2 in Table 4), and
2. uncertain travel times (see Type 3 in Table 4).

The first stochastic extension is from the perspective of travelers' perception (or the demand side) and is based on the fact that travelers may not have entirely accurate traffic information or sufficient knowledge of the actual traffic conditions. The second extension is from the perspective of stochastic road networks (or the supply side) and is based on the fact that travel times are uncertain due to the effects of random events, such as traffic incidents and traffic signal failure.

2.2.1 Travelers' perceptions of travel time/cost

One example of the extension of the first dimension is the principle of *stochastic user equilibrium* (SUE) [23]. According to this principle, travelers are assumed to choose their routes based on perceived travel times, rather than the actual travel time, where each traveler may perceive a different travel time for the same link. Perceived travel time is defined as the sum of expected travel time and the perception error, where the perception error is modeled by a probability distribution to capture the variation in the perception of travel time. SUE is reached when no traveler can improve his/her perceived travel time by unilaterally changing routes. SUE is more general than UE. If perceived travel times are assumed to be entirely accurate, all travelers will perceive the same travel time over the same link and SUE will be identical to UE.

2.2.2 Uncertain travel times

Regarding uncertain travel time, the following eight equilibrium concepts have been defined:

- risk user equilibrium,
- risk system optimum,
- mean excess traffic equilibrium (METE),
- reliability-based user equilibrium (RBUE),
- percentile equilibrium,
- risk-averse user equilibrium (RAUE),
- robust user equilibrium, and
- prospect-based user equilibrium.

They have different behavioral assumptions that need to be verified for the dataset used. In general, they can be classified into five categories, namely (1) safety margin, (2) travel time reliability (3) travel time in the worst case scenario, (4) percentile travel time and (5) travel prospect value.

(1) Safety margin

The safety margin is the extra time reserved to increase the chance of reaching a destination on time. This concept considers the fact that travelers can depart earlier to avoid late arrivals. This concept has been incorporated by risk user equilibrium, risk system optimum and METE.

Risk user equilibrium is an extension of UE and is developed based on effective travel time [24] or travel time budget [25] instead of travel time as in UE. Effective travel time is defined as the sum of expected travel time and a safety margin. In this case, risk user equilibrium is reached when all used routes between the same OD pair have the same travel time budget [26]. Uchida and Iida [26] modeled the safety margin of a route as the product of the standard deviation of route travel time and a parameter representing the degree of risk aversion of drivers. A larger parameter value means that a driver is more risk-averse. When the parameter equals zero, a driver is risk neutral and ignores the variability of travel time. Jackson and Jucker [27] mentioned that in Brastow and Jucker [28] the safety margin can also be modeled using the variance of travel time instead of the standard deviation. This concept has been verified by many authors such as Jackson and Jucker [27], Lo et al. [25], and Szeto et al. [29].

Risk system optimum proposed by Uchida and Iida [26] is an extension of SO. Risk system optimum is defined by the sum of the travel time budget instead of total travel time as in SO. The first order condition can be defined by the "marginal" travel time budget, which is the derivative of the total travel time budget with respect to flow. This differs from of risk user equilibrium that defines the first order conditions by travel time budget. Risk system optimum is useful in planning large traffic studies, where traffic management techniques are used to discourage or encourage traffic under the travel time uncertainty. This principle is also useful for depicting logistic flow when the transport time is uncertain.

METE is defined by the mean excess travel time (METT) [30]. Different from risk user equilibrium, the METE considers the penalty of being late in addition to expected travel time and the safety margin. METT is equal to the travel time budget plus the expected excess travel delay, in which the expected excess travel delay reflects the penalty of being late given the travel time budget selected. METE is said to be reached when all used routes between each OD pair have equal METT, and no unused route has a lower METT. However, METT has not been verified from the empirical data satisfactory (see Franklin and Karlstrom [31] for details).

(2) Reliability of travel time

Reliability of travel time has been directly incorporated into the equilibrium concept proposed by Chan and Lam [32] called *reliability-based user equilibrium* (RBUE). This concept is characterized by a normalized path preference index (PI) instead of (path) travel time as in UE. The PI is defined as the weighted sum of the path travel time index (TI) and path travel time reliability index (RI). TI is defined as the value of a monotonously decreasing, non-negative exponential function of path travel time with a largest value of 100, in which the largest value occurs when the travel time of the path is equal to the free-flow path travel time (i.e., the minimum travel time required to traverse the path when there is no flow on the path). The RI equals path travel time reliability multiplied by 100, and path travel time reliability is defined as the probability that the actual path travel time is not larger than the acceptable travel time. The sum of the weights on the two indices is one and a larger weight is associated with a more important index. For a more risk-averse traveler, a larger weight is associated with RI whereas for a risk-neural traveler, the weight on RI equals 0. For the latter, RBUE becomes

UE, and at equilibrium, all used paths have the same PI, implying that all used paths have the same travel time. To the best of our knowledge, this concept has not been verified by empirical data.

(3) Percentile travel time

A percentile of travel time is the travel time below which a certain percent of travel time can be found. A specified percentile of travel time (instead of mean travel time as in UE) was proposed by Ordóñez and Stier-Moses [33] to define *percentile equilibrium*, and is used to reflect the risk-averse behavior of travelers. A larger percentile means that travelers are more risk-averse. In the special case, when travelers are risk neutral, the percentile is the 50^{th} percentile travel time. In this equilibrium concept, travelers are assumed to choose routes that minimize a specified percentile of travel time. At equilibrium, the percentile of travel time is equal for all travelers between the same OD pair. This percentile equilibrium seems to be a better representation of the reality than risk user equilibrium since Lam and Small [34] found that percentile is a better measure of reliability than the standard deviation in a route choice experiment. However, percentile equilibrium is more difficult to be computed since it involves convolution of link travel time distributions.

(4) Travel time in the worst case scenario

Two equilibrium concepts were proposed based on worst case scenarios, including *risk-averse user equilibrium* and *robust user equilibrium.* The *risk-averse user equilibrium*, proposed by Bell and Cassir [35], is based on a Nash game framework in which there are two types of players, namely travelers and demons. Travelers aim to minimize the expected travel time and demons aim to maximize the total travel time of all travelers. In this game, it is assumed that there is only one demon per OD pair and the number of travelers is fixed. The assumption of one demon per OD pair was relaxed by Szeto et al. [36,37], where there can be any number of demons in a network but they are non-cooperative. Szeto [38] further relaxed the non-cooperative assumption so that all demons are cooperative. On the other hand, Szeto et al. [39] also relaxed the assumption of a fixed number of travelers. The travel time in the game theory framework can be considered as the expected travel time in the worst case scenario defined by the number of demons and the degree of cooperation.

Robust user equilibrium is defined based on the worst-case travel time [40]. The worst-case travel time (of a path) is computed assuming that the number of arcs (on the path) along with their maximum travel times does not exceed the budget of uncertainty. The budget of uncertainty is a parameter associated with every driver and represents his/her degree of risk aversion. A higher budget implies a more risk-averse driver. This version of robust user equilibrium differs from the one proposed by Zhang et al. [41], which is for forecasting from the viewpoint of transport network planners, not for modeling the route choice behavior of drivers. This version is also different from risk averse user equilibrium in the sense that robust user equilibrium considers worst-case travel time deviation for “every” path. The robust user equilibrium and risk user equilibrium were viewed as two different approximations of percentile equilibrium [40].

(5) Travel prospect value

Travel prospect value, which is derived from the cumulative prospect theory proposed by Tversky and Kahneman [42], is a weighed expected utility, and is defined by a nonlinear function that can depict the following three behavioral principles observed in many experiments:

(i) People distinguish gains from losses before making choices and the payoffs are framed as gains or losses with respect to some reference points [e.g., 43].

(ii) The losses looms larger than the gains, i.e., people generally care more about potential losses than potential gains. At the same time, they are risk-averse in regard to gains and risk-seeking in regard to losses [e.g., 44].

(iii) People tend to overweight the significance of extreme but unlikely events. At the same time, they underweight "average" events [e.g., 45].

Travel prospect value was validated by empirical experiments [e.g., 46] and used to define *prospect-based user equilibrium* [47]. A prospect-based user equilibrium is said to be achieved when no traveler can improve his or her travel prospect value by unilaterally changing his/her route.

2.2.3 Simultaneous consideration of two stochastic extensions

A number of equilibrium principles have been extended from Wardrop's first principle in both stochastic dimensions (see Type 4 in Table 4). For example, the generalized traffic equilibrium proposed by Mirchandani and Soroush [48] considers travelers' perception errors and probabilistic travel times. The route choice criterion is to minimize the perceived expected disutility, which is a function of the random travel time. The traveling risk is implicitly included in the disutility function. Another example is the *reliability-based stochastic user equilibrium* (RSUE) proposed by Shao et al. [49] which is extended from risk user equilibrium using perceived travel time budget to define RSUE. The third example is s*tochastic METE* proposed by Chen and Zhou [50], which is an extension from METE where perceived METT is used instead of METT.

2.3 Dynamic extensions of Wardrop's principles

Compared with static traffic assignment, travelers have one additional consideration, which is departure time, no matter it is fixed or not. Hence, many studies focused on the choice of departure time (see [51-54] for the literature) and all the dynamic extensions capture the departure time consideration.

2.3.1 Simple dynamic extensions

The following equilibrium principles, referred to as Type 5 principles in Table 4, for the travel choice component of DTA can be considered as simple dynamic extensions of the travel choice principles adopted in static traffic assignment:

- the dynamic user equilibrium (DUE) or dynamic user optimal (DUO) route choice principle [55];
- the DUE departure time choice principle [56];
- the DUE route/departure time choice principle [57]; and
- the dynamic system optimal (DSO) principle [58].

The dynamic user equilibrium (DUE) route choice principle, which is the simplest dynamic extension of Wardrop's [21] first principle, states that for each origin-destination pair, any routes used by travelers *departing at the same time* must have equal and minimal travel time. This principle is used when the demand at each departure time is known. That is, the principle is often used in the pure dynamic route choice problem.

The DUE departure time choice principle considers departure time choice instead of route choice. This principle requires that the generalized travel costs for travelers between the same OD pair departing at *any* time are equal and minimal. The principle is adopted when the route choice is pre-determined or there is no route choice for travelers (i.e., the pure departure time choice problem) and the generalized travel cost normally includes the penalty cost due to early and/or late arrivals in addition to the travel time cost.

The DUE route/departure time choice principle considers departure time choice in addition to route choice, and considers generalized travel cost instead of travel time. This principle states that for each OD pair, the generalized travel costs incurred by travelers departing at any time using any route are equal and minimal. This principle is essentially a generalization of the DUE route choice principle and the DUE departure time choice principle, and can be used in the simultaneous route and departure time choice problem. In fact, the example discussed in the introduction is constructed using this principle. Assuming that 1 minute of travel time is equivalent to 1 dollar and the cost of early arrival is 0.5 dollar per minute, the minimum generalized travel cost is equal to 90 dollars for all drivers. The travel time cost for the last driver is 90 dollars as the travel time is 90 minutes. The penalty cost for the last driver is 0 dollar, as he/she arrived at B at 9:00 am sharp. The travel time cost for the first driver is 30 dollars and the penalty cost is 60 dollars (i.e., 120 minutes multiplied by 0.5 dollar per minute). If this driver chose Link 1 instead, he/she would arrive at B at 7:30 am, and his/her travel time cost and penalty cost would become 60 dollars and 45 dollars, respectively, leading to a general travel cost of 105 dollars. Therefore, the choice of departing at 6:30 am and going to B via Link 1 is not optimal and no driver picks this choice at equilibrium.

The DSO principle is an extension of Wardrop's second principle, and assumes that each traveler chooses his/her route cooperatively with other travelers to ensure the total system travel time over the modeling horizon is minimized.

2.3.2 Stochastic extensions of DUE

Examples can also be found based on the stochastic extension of DUE (See Type 6 in Table 4). The *stochastic dynamic user equilibrium (SDUE) for departure time choice* [59] is the stochastic extension of DUE for departure time choice and considers perceived generalized travel cost instead of generalized travel cost. *The SDUE route choice principle* proposed by Ran and Boyce [60] is the stochastic extension of the DUE route choice principle and considers perceived travel time instead of actual travel time. This principle can also be considered as the dynamic extension of the SUE route choice principle. Similarly, *the SDUE route/departure time choice principle* proposed by Vythoulkas [61] is a generalization of the DUE route/departure time choice principle and considers perceived generalized travel cost instead of actual generalized travel cost.

There are a number of dynamic equilibrium concepts that consider both perception error and uncertain travel time (see Type 7 in Table 4). For example, Boyce et al. [62] considered the dynamic extension of the generalized traffic equilibrium. The *reliability-based stochastic dynamic user equilibrium (RSDUE) route choice principle* proposed by Szeto et al. [63] is extended from the RSUE principle, and considers the attitudes of travelers towards the risk of late arrivals due to uncertain travel times, in addition to variations in their perception of the travel times. Travelers are assumed to select routes with the lowest perceived effective travel times. This RSDUE principle states that for each traveler departing at the same time, the perceived effective travel time of their routes must be equal and minimal.

2.4 Bounded rationality-based principle

The last type in Table 4 is Type 8, which is related to the extension of UE based on bounded rationality. The term ‘bounded rationality’ is used to describe rational choices that take into account the limitations of the decision-maker in terms of knowledge, computational capacity and time to make decisions [64]. This term also refers to the rational principles that underlie the non-optimizing adaptive behaviors of real people. The idea was originally proposed [65] and refined by Simon [66]. Based on this idea, Mahmassani and Chang [67] proposed the concept of a *boundedly rational user equilibrium* (BRUE), in which they analyze in relation to the departure time choice problem. The assumption is that travelers with bounded rationality tend to maximize their individual utility, but not necessarily to an absolute maximum level. Szeto and Lo [68] considered this concept in their dynamic traffic assignment problem with route choice only. Lou et al. [69] considered it in regard to congestion pricing, and defined BRUE as follows:

> “Travelers with bounded rationality are those who (a) always choose routes with no cycle and (b) do not necessarily switch to the shortest (cheapest) routes when the difference between the travel times (or costs) on their current routes and the shortest one is not larger than a threshold value.”

2.5 Trends and observations

Although the general trend is that the travel choice principles or the equilibrium concepts in DTA are extended from those in static traffic assignment, not all the equilibrium concepts in static traffic assignment have been extended. From Table 4, we can see that some of the travel choice equilibrium concepts used in static traffic assignment, such as risk system optimum, METE, percentile equilibrium, robust user equilibrium, RBUE, risk-averse user equilibrium, and stochastic METE, have not been extended to DTA. It is not surprising that percentile equilibrium, robust user equilibrium, METE, and stochastic METE have not been extended to DTA, given that these are relatively new concepts. They may be currently investigated and validated by empirical data. However, the risk system optimum, RBUE, and risk-averse user equilibrium have been proposed for some time. They deserve to be extended to DTA and validated empirically. Overall, the equilibrium concepts that have not been extended to DTA can be improved in future studies and validated using real data.

Another observation is that the recent (dynamic) travel choice components developed become more and more complicated to capture more and more realistic travel behaviors. However, to the best of our knowledge, the more realistic behavioral principles often introduce the convergent issues during the computation process when they are combined with even simple traffic flow component, since the resulting models do not satisfy the convergence requirement of existing solution algorithms. This implies that the existing DTA models become more and more complicated, and more and more difficult to solve.

3 Dynamic traffic assignment: Model classifications

The example given in Section 1 belongs to one class of DTA problems where route and departure time choices are both considered and the travel demand is fixed. In general, DTA can be modeled in different ways, and the DTA models can be classified based on the criteria such as choice dimension modeling, overall formulation approaches, and time dimension modeling, as shown in Table 5. These criteria are further divided into sub-criteria.

Table 5 Criteria for classifying DTA models

Criteria	Sub-criteria	Categories
Choice dimension modeling	Route and departure time choice	• Pure departure time choice [e.g., 56,59] • Pure route choice o En route adjustment/reactive [e.g., 71-73] o no en route adjustment/predictive [e.g., 55,63,68,74,75]. • Route and departure time choices [e.g., 57,70].
	Whether travelers must travel or not (or demand elasticity)	• Fixed demand [e.g., 76] • Elastic demand [e.g., 70].
Time dimension modeling	Duration of the study horizon	• Day-to-day [e.g., 73,77,78] • Within-day [e.g., 60,79,80].
	Study horizon modeling	• Continuous [e.g., 55,80] • Discrete [e.g., 65,68].
Overall formulation approaches	Decision variable (or formulation)	• Link flow (or link based) [e.g., 80-83] • Route flow (or route based) [e.g., 63,75,84]
	Queue representation	• Physical queue [e.g., 63,75,76,84] • Non-physical queue [e.g., 60, 80-83]
	Number of classes of travelers	• Single class [e.g., 68,75] • Multi-class [e.g., 63,86]
	Methodological approaches	• Simulation [e.g., 87,88] • Analytical o NCP [e.g.,70], o VIP [e.g., 84], o MPP [e.g., 75], o FPP [e.g., 63] o OCP [e.g., 55] o CMP [e.g., 89-99].

3.1 Choice dimension modeling

3.1.1 Route and departure time choices available

The first way of classifying DTA models is based on the route and departure time choices available. In general, DTA models can be classified into (1) the pure departure time choice model [e.g., 56,59], (2) the pure route choice model, and (3) the simultaneous route and departure time choice model [e.g., 57,70]. The pure departure time choice model only considers departure time choice and drivers have no alternative routes to their destinations. That is, drivers can decide to leave earlier or later to avoid congestion and to arrive at their destination on time. The pure route choice model assumes that the departure times are fixed, but each driver has a choice of routes. The simultaneous route and departure time choice model considers both route and departure time choices, and includes the pure route and the pure departure time choice models as two special cases. These DTA models can be further classified by the travel choice principle adopted (See section 2), such as the DUE route choice model and the SDUE route choice model.

For the pure route choice model, it can be further classified into the *en route* adjustment model or the reactive DTA model [e.g., 71-73] and no *en route* adjustment model or the predictive DTA model [e.g., 55,63,68,74,75]. The *en route* adjustment model allows drivers to switch their routes during their trips in response to having more update traffic information. For example, a driver will switch to another route if he/she realizes that there is a heavy traffic jam in front of his/her originally planned route. This adjustment model contrasts with the no *en route* adjustment model, which assumes that choices do not change during trips and that travelers select routes based on pre-trip information and predicted travel times. The *en route* adjustment model has not been combined with a departure time choice model to capture both departure time and *en route* choices.

3.1.2 Demand elasticity

DTA models can be roughly classified into the DTA model with fixed (or inelastic) demand [e.g., 76] and the DTA model with elastic demand [e.g., 70]. For the elastic demand case, travelers can decide to give up their trips or decide to use public transport. This consideration extends DTA models to capture mode choice and the choice of not making a trip. For the fixed demand case, travelers have to make a trip by private cars.

3.2 Time dimension modeling

3.2.1 Duration of the study horizon

DTA models can be roughly divided into the day-to-day adjustment model [e.g., 73,77,78] and the within-day DTA model [e.g., 60,79,80]. The day-to-day adjustment model is concerned with how the travel decisions of travelers change over the day and how their route or departure time choices on a particular day depend on their experience obtained in previous days. The within-day model includes the pure departure time choice model, the pure route choice model, and the simultaneous route and departure time choice model, in the sense that the travel decision is considered in a typical day and there is no day-to-day adjustment. The solution of the within-day model can also be considered as the final state solution of the day-to-day adjustment model, where no traveler can make a better decision than their current one.

3.2.2 Study horizon modeling

In DTA models, the study horizon can be modeled in continuous time settings [e.g., 55,80] and discrete time settings [e.g., 65,68]. Hence, DTA models can be divided into the continuous-time DTA model and the discrete-time DTA model. However, to solve the continuous-time DTA model, it is normal to first discretize the modeling horizon. Therefore, formulating DTA in a continuous time setting is only conducted for the purpose of accurately modeling the problem.

3.3 Overall formulation approaches

3.3.1 Decision variables

Analytical DTA models are link-based [e.g., 80-83] or route-based [e.g., 63,75,84], if the decision variables are link flow or route flow, respectively. The advantage of using link flow variables is that path enumeration and a complete route set are not needed during the solution process. However, queue spillback (i.e., queues spilling backwards to upstream links) cannot be captured by the link-based models because there is no information to direct queues spilling backwards to specified upstream links. If queue spillback is the key concern of DTA modeling, a route-based model should be used, as the route information can direct queues spilling backward to specified upstream links.

3.3.2 Queue modeling

DTA models can broadly be classified as the physical-queue (or spatial-queue) DTA model [e.g., 63,75,76,84] or the non-physical queue DTA model [e.g., 60, 80-83], depending on the traffic flow model adopted. The physical-queue traffic flow model [e.g., 85] considers the length of vehicles, and emphasizes the effects of queue spillback and junction blockage. To model how a queue spills backwards, the physical-queue DTA model is always formulated as a route-based model. On the other hand, the non-physical queue DTA model ignores the vehicle length and hence queue spillback cannot be modeled properly. The non-physical queue DTA model can be formulated as either a link-based or a route-based model depending on the solution method used for solving for solutions. More model examples and detailed discussions of these two queuing approaches can be found in [16,17].

3.3.3 Number of classes of travelers

DTA models can be divided into the single-class DTA model [e.g., 68,75] and the multi-class DTA model [e.g., 63,86]. The introduction of multiple classes allows the behavior of travelers with different values of travel time and the different unit costs of early and late arrivals to be captured. The extension of travel choice principles to multiple classes is straightforward. For example, the DUE route choice principle can be revised as follows: *For each class of traveler* and for each origin-destination pair, any routes used by travelers departing at the same time must have equal and minimal travel time. The introduction of multiple classes also allows each class of driver to have their own travel choice principle. This feature has been used in route planning and guidance services [86], where both the DUE and SDUE route choice conditions have been used.

3.3.4 Methodological approaches

One way to classify DTA models is according to whether the model development approaches are (1) simulation-based [e.g., 87,88] or (2) analytical-based.

(1) Simulation-based

The simulation-based approaches emphasize microscopic traffic flow characteristics, such as lane changing. Strict adherence to travel choice principles, such as Wardrop's principle, is secondary. Earlier generations of this approach used intersection-turning ratios to split traffic without route specification. Some models specify route choices based on the k-shortest routes criteria, which is then extended to the concept of "bounded rationality" for dynamic route switching [67]. This approach shares the following properties. First, the models are essentially descriptive, not prescriptive tools. They simulate the probable results of certain traffic management strategies, but do not prescribe what a strategy ought to be. Second, they lack well-defined solution properties. One cannot prove whether a solution has achieved the required optimality. In each computer simulation, the model produces a realization out of a large range of possible realizations. Therefore, one must be careful in generalizing or transferring the results.

(2) Analytical-based

The analytical-based approaches normally consider macroscopic traffic behavior and have well-defined properties, in terms of optimality conditions and adherence to dynamic versions of Wardrop's principle [21]. Depending on how the model is developed, these models may be used for prescriptive or descriptive purposes. The main difficulty with the analytical approaches is adding realistic traffic dynamics, such as queue spillback, to already complicated formulations. Therefore, some studies ignore the effects of queue spillback when developing analytical DTA models.

The analytical approaches often express DTA as one of the following problems:

- the Nonlinear Complementarity Problem (NCP) [e.g.,70],
- the Variational Inequality Problem (VIP) [e.g., 84],
- the Mathematical Programming Problem (MPP) [e.g., 75],
- the Fixed-Point Problem (FPP) [e.g., 63],
- the Optimal Control Problem (OCP) [e.g., 55] and,
- the Continuum Modeling Problem (CMP) [e.g., 89-99].

The NCP is to find $\mathbf{f}^* = \left[f_p^{rs*}(t) \right]$ such that:

$$\mathbf{f}^{*\mathrm{T}} \cdot \mathbf{H}\left(\mathbf{f}^*\right) = 0;\ \mathbf{H}\left(\mathbf{f}^*\right) \geq \mathbf{0};\ \text{and } \mathbf{f}^* \geq \mathbf{0}, \tag{4}$$

where $\mathbf{f}$ and $f_p^{rs}(t)$ are, respectively, the vector of route flows and the flow departing at origin r at time t and the travel on route p between OD pair rs. The superscript "*" refers to a solution of $\mathbf{f}$ that fulfills the traffic equilibrium conditions. $\mathbf{H}(\mathbf{f})$ represents a general vector function of $\mathbf{f}$. For the case of user equilibrium DTA where the demand is a decreasing function of minimum travel time, $\mathbf{H}(\mathbf{f}) = \left[n_p^{rs}(t) - \pi^{rs}(t) \right]$, in which $n_p^{rs}(t)$ is the route travel time for travelers departing at origin r at time t and going to destination s via route p. This NCP formulation allows us to easily observe how the travel choice principle and the traffic flow component are integrated. In fact, by comparing (1)-(3) with (4), we can conclude that the travel choice principle is expressed as a NCP. Meanwhile, the route travel time $\mathbf{n} = \left[n_p^{rs}(t) \right]$ in $\mathbf{H}(\mathbf{f})$ represents the traffic flow component and can be written as $\mathbf{n} = \mathbf{\Phi}(\mathbf{f})$ where $\mathbf{f}$ is the vector of route flows, and $\mathbf{\Phi}(\mathbf{f})$ is a unique travel time mapping of route flows via a dynamic traffic flow model.

The DTA problem can also be formulated as a VIP, which includes the NCP as a special case (Proposition 1.4 in [100]). The VIP can be expressed as to find $\mathbf{f}^* = \left[f_p^{rs*}(t) \right]$ such that:

$$\left(\mathbf{f} - \mathbf{f}^*\right)^{\mathrm{T}} \mathbf{H}\left(\mathbf{f}^*\right) \geq 0, \forall \mathbf{f} \in \Omega, \tag{5}$$

where Ω is a closed convex set. For the case of user equilibrium DTA with fixed demand, $\mathbf{H}(\mathbf{f}) = \left[n_p^{rs}(t) - \pi^{rs}(t) \right]$. Ω is the feasible solution set of the problem defined by $\Omega = \left\{ f_p^{rs}(t) \mid q^{rs}(t) = \sum_p f_p^{rs}(t), \forall rs, t,\ f_p^{rs}(t) \geq 0, \forall p, rs, t \right\}$ where $q^{rs}(t)$ is the demand between OD pair *rs* departing from the origin at time *t*. This set is formed by the dynamic extension of the non-negativity condition (1) and the flow conservation condition to ensure that the total route flow departing at time *t* is equal to the corresponding demand. The existence of solutions to the VIP requires that (i) $\mathbf{H}(\mathbf{f})$ is a continuous function of $\mathbf{f}$ and (ii) Ω is a nonempty compact convex set (Theorem 1.4 in [100]). The uniqueness of the solution further requires the mapping function to be strictly monotonic (Theorem 1.8 in [100]). Most of DTA papers formulate the problem as a VIP because the advantages of adopting the VIP approach for modeling and analyzing general DTA problems [14]. The equivalency conditions of the VIP (5) and the NCP (4) are also discussed in Proposition 1.4 of Nagurney [100], which states that the solutions to these problems are equivalent when the feasible solution region is the non-negative orthant.

The MPP can be obtained from a NCP via a gap function for the NCP. The function $G\colon R_+^n \to R_+^1$ is a gap function for the NCP if the following three conditions are satisfied [75]:

i. $G(\mathbf{f}) \geq 0$,
ii. $G(\mathbf{f}) = 0 \Leftrightarrow \mathbf{f} \in \Omega'$,
iii. $\min_{\mathbf{f}\in\Psi} G(\mathbf{f}) = 0$ is a global minimum,

where Ω' be the set of solutions to the NCP, and $\Psi = \{\mathbf{H}(\mathbf{f}) \geq \mathbf{0}, \mathbf{f} \geq \mathbf{0}\}$. One example of the gap function is $G(\mathbf{f}) = \frac{1}{2}\sum_t \sum_{rs} \sum_p \left[\sqrt{f_p^{rs}(t)^2 + H_{p,t}^{rs}(\mathbf{f})^2} - f_p^{rs}(t) - H_{p,t}^{rs}(\mathbf{f})\right]^2$, where $\mathbf{H}(\mathbf{f}) = \left[H_{p,t}^{rs}(\mathbf{f})\right]$. This gap is used in Lo and Szeto [75] in formulating the DUE DTA problem. The MPP is then described as:

$$\min_{\mathbf{f}\in\Psi} G(\mathbf{f}) \tag{6}$$

When the travel choice condition is satisfied, the objective value $G(\mathbf{f})$ is equal to zero. Otherwise, $G(\mathbf{f})$ is always greater than zero.

The FPP is to find $\mathbf{f}^* = \left[f_p^{rs*}(t)\right]$ such that:

$$\mathbf{f}^* = \mathbf{Y}(\mathbf{f}^*), \tag{7}$$

where $\mathbf{Y}(\mathbf{f})$ represents a general vector function of $\mathbf{f}$. If $\mathbf{Y}(\mathbf{f}) = P_\Omega(\mathbf{f} - \kappa\mathbf{H}(\mathbf{f}))$, where the projection operator $P_\Omega(\mathbf{y}) = \arg\min_{\mathbf{z}\in\Omega} \|\mathbf{y} - \mathbf{z}\|$, $\kappa > 0$, Ω is closed and convex, then the FPP (7) and the VIP (5) have the same set of optimal solutions (Theorem 1.3 in [100]).

DTA can be formulated as an OCP [e.g., 55], which can be viewed a special case of an optimization problem in which the objective function involves an integral of a function over time and the constraints involve partial differential equations. This approach does not receive much attention for modeling user equilibrium DTA nowadays because of the inherent limitation for modeling traffic propagation and cannot handle multiple OD pairs for some cases (See e.g. [15]). However, the OCP approach is still used for modeling and analyzing the DSO problem [101,102].

The continuum modeling approach has also been used to model DTA problems for pedestrian and urban traffic flows, in which the system is represented by a two-dimensional continuum transportation system. The problems are formulated as a system of partial differential equations, and solved by finite element or finite difference numerical schemes [e.g., 89-99]. Currently, only pure route choice problem has been considered.

3.4 Discussion

From the above discussion, it is not difficult to see that all travel choice principles mentioned in Section 2 can be expressed in NCP format, which can be further expressed as VIP, MPP and FPP formats, because the equivalent conditions always hold. Moreover, the traffic flow component is expressed as a mapping of $\mathbf{H}(\mathbf{f})$ in these problem formulations.

The solution existence and uniqueness depend on the properties of $\mathbf{H}(\mathbf{f})$ in NCP, VIP, MPP and FPP, which in turn depend on both the travel choice principle and the traffic flow model adopted. If a physical-queue traffic flow model or a discontinuous travel choice model (due to for example a stepwise penalty cost function) is chosen, solution existence may not be guaranteed [68]. If the travel choice principle and the traffic flow model selected cannot make $\mathbf{H}(\mathbf{f})$ strictly monotonic, there may be multiple solutions. For other solution properties, one can refer to [16,17,103].

If the travel choice principle and the traffic flow model selected do not lead to a nice property of $\mathbf{H}(\mathbf{f})$ required by existing solution algorithms (e.g., monotonicity, pseudomonotonicity, co-coercivity and Lipschitz continuity), the error or gap measuring the distance between the optimal solution and the current solution will not decrease monotonicity and in some cases, convergence may not be achieved. This is often the case for complicated models developed to capture realistic traffic and travel behaviors. For the solution methods of each of the formulations, one can refer to [63,68,70,75,84] for further details.

There are many classifications of DTA models, reflecting the multiple aspects of DTA modeling, including realistic representation, solution efficiency and ease of analyzing the problem. For example, in terms of queue representation, some publications focus on modeling realistic queue spillback. Hence, they simply classify DTA models into physical queue and non-physical queue models, in which only physical queue DTA models can capture queue spillback. However, physical queue DTA models are usually more complicated than point queue DTA models so as to capture queue spillback, and more difficult to solve efficiently because $\mathbf{H}(\mathbf{f})$ does not have nice properties, such as monotonicity, pseudomonotonicity, co-coercivity, and Lipschitz continuity, required by existing algorithms. In terms of the choice consideration, pure departure time choice models are easier to analyze and obtain optimal solutions but are less realistic as compared with DTA models that consider both route and departure time choice. In terms of analytical approaches, the choice of formulations highly depends on the solution method adopted in addition to analyzing the condition of solution existence and uniqueness. For instance, a VIP formulation will be used when projection methods will be adopted for obtaining solutions and the mathematical properties of $\mathbf{H}(\mathbf{f})$ will be examined. It is worth noting that the classification used in each DTA publication depends on the focus of the publication. Therefore, there are various model classifications.

4 Concluding Remarks

This paper serves as a supplementary publication to the existing reviews by reviewing the current travel choice principles adopted in static traffic assignment and DTA as well as the classifications of DTA models. How the travel choice principle is combined with the traffic flow component is also discussed using the NCP, VI, MPP and FPP formulations. This paper also points out that there are many classifications of DTA models, reflecting the multiple aspects of DTA modeling, including realistic representation, solution efficiency and ease of analyzing the problem. Some observations and model limitations are then discussed. In particular, we also observe that the travel choice principles or the equilibrium concepts in DTA are extended from those in static traffic assignment. Moreover, the existing DTA models capture more and more realistic traffic or travel behavior, and hence become more and more complicated, and more and more difficult to solve. We also find that the OCP approach is currently used for modeling and analyzing the DSO problem only where as the VIP approach received most attention in the DTA literature. We expect that this trend will continue because the advantages of adopting the VIP approach for modeling and analyzing general DTA problems.

From this review, we can identify three future research directions. First, some existing travel choice principles such as reliability-based user equilibrium and SMETE have not been verified empirically. Therefore, one direction is to verify these principles before extending them to the dynamic case. Second, a number of travel choice equilibrium concepts, such as the risk system optimum, percentile equilibrium, robust user equilibrium, and risk-averse user equilibrium, have not been extended to DTA. For except risk-averse user equilibrium, the simplest dynamic extension of these equilibriums is to introduce those equilibrium conditions for each departure time choice. For risk-averse user equilibrium, one can consider demons to select links to damage throughout the modeling period. Third, there is a need to develop solution methods with a looser convergence condition of $\mathbf{H}(\mathbf{f})$ to solve complicated DTA models. Hence, developing convergent solution algorithms must be one future direction. Fourth, the CMP approach was only used for studying the pure route choice DTA problem. This approach is pity new and has not been mature yet. A lot of research can be done on this approach. One future direction for the CMP approach can be to extend the current work to model the simultaneous route and departure time problem and the multi-class DTA problem. Finally, the *en route* adjustment model has not been combined with a departure time choice model to capture both departure time and *en route* choices. Analyzing en route adjustment together with departure time consideration is one potential future research area.

Acknowledgements

The research was jointly supported by a grant from the Hui Oi Chow Trust Fund (200902172003), two grants from the University Research Committee of the University of Hong Kong (201001159008 and 201011159026), a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (HKU 7183/08E), and Arup Global Research Challenge Award 2011. We are grateful to the three anonymous referees for their constructive and valuable comments.

References

1. Gomes G., Horowitz R., Optimal freeway ramp metering using the asymmetric cell transmission model, TRANSPORT RES C-EMER., 2006, 14, 244-262.
2. Meng Q., Khoo H.L., A Pareto-optimization approach for a fair ramp metering, TRANSPORT RES C-EMER., 2010, 18, 489-506.
3. Park B.B., Yun I., Ahn K., Stochastic optimization for sustainable traffic signal

control, INT J SUSTAIN TRANSP., 2009, 3, 263-284.
4. Park B.B., Kamarajugadda A., Development and evaluation of a stochastic traffic signal optimization method, INT J SUSTAIN TRANSP., 2007, 1, 193-207.
5. Lo H.K., Szeto W.Y., Modeling advanced traveler information services: static versus dynamic paradigms, TRANSPORT RES B-METH., 2004, 38, 495-515.
6. Szeto W.Y., Lo H.K., The impact of advanced traveler information services on travel time and schedule delay costs, J INTELL TRANSPORT S., 2005, 9, 47-55.
7. Lo H.K., Szeto W.Y., Road pricing modeling for hyper-congestion, TRANSPORT RES A-POL., 2005, 39, 705-722.
8. Zhong R.X., Sumalee A., Maruyama T., Dynamic marginal cost, access control, and pollution charge: a comparison of bottleneck and whole link models, J ADV TRANSPORT., 2011, Accepted.
9. Ukkusuri S., Waller S., Linear programming models for the user and system optimal dynamic network design problem: formulations, comparisons and extensions, NETW SPAT ECON., 2008, 8, 383-406.
10. Szeto W.Y., Ghosh, B., Basu, B., O'Mahony M., Cell-based short-term traffic flow forecasting using time series modelling, TRANSPORT ENG-J ASCE., 2009, 135, 658-667.
11. Abdelghany A.F., Adbelghany K.F., Mahmassani H.S., Murray P.M., Dynamic traffic assignment in design and evaluation of high-occupancy toll lanes, TRANSPORT RES REC., 2000, 1733, 39-48.
12. Yagar S., Dynamic traffic assignment by individual path minimization and queuing, TRANSPORT RES., 1971, 5, 179-196.
13. Cascetta E., Cantarella G.E., Modelling dynamics in transportation networks: state of the art and future developments, SIMULAT PRACT THEORY., 1993, 15, 65-91.
14. Peeta S., Ziliaskopoulos A.K., Foundations of dynamic traffic assignment: the past, the present and the future, NETW SPAT ECON., 2001, 1, 233-265.
15. Boyce D., Lee D.H., Ran B., Analytical models of the dynamic traffic assignment problem, NETW SPAT ECON., 2001, 1, 377-390.
16. Szeto W.Y., Lo H.K., Dynamic traffic assignment: review and future research directions, JOURNAL OF TRANSPORTATION SYSTEMS ENGINEERING AND INFORMATION TECHNOLOGY, 2005, 5, 85-100.
17. Szeto W.Y., Lo H.K., Properties of dynamic traffic assignment with physical queues, J EAST ASIA SOC TRANSPORT STUD., 2005, 6, 2108-2123.
18. Mun J.S., Traffic performance models for dynamic traffic assignment: an assessment of existing models, TRANSPORT REV., 2007, 27, 231-249.
19. Jeihani M., A review of dynamic traffic assignment computer packages, JOURNAL OF THE TRANSPORTATION RESEARCH FORUM, 2007, 46, 35-46.
20. Szeto W.Y., Cell-based dynamic equilibrium models, dynamic traffic assignment and signal control, In: Dynamic Route Guidance and Traffic Control, Complex Social, Economic and Engineered Networks Series, Ukkusuri S., Ozbay K. (Eds.), Springer, 2011, submitted.
21. Wardrop J., Some theoretical aspects of road traffic research, ICE Proceedings: Part II, Engineering Divisions, 1952, 1, 325-362.
22. Oppenheim N., Urban Travel Demand Modelling: From Individual Choice to General Equilibrium, John Wiley & Sons, USA, 1995.
23. Daganzo C.F., Sheffi Y., On stochastic models of traffic assignment, TRANSPORT SCI., 1977, 11, 253-274.
24. Hall R.W., Travel outcome and performance: the effect of uncertainty on accessibility, TRANSPORT RES B-METH., 1993, 17, 275-290.

25. Lo H.K., Luo X.W., Siu B.W.Y., Degradable transport network: travel time budget of travellers with heterogeneous risk aversion, TRANSPORT RES B-METH., 2006, 40, 792-806.
26. Uchida T., Iida Y., Risk assignment: a new traffic assignment model considering the risk travel time variation, In: Daganzo C.F. (Ed.), Proceedings of the 12th International Symposium on Transportation and Traffic Theory (July 21-23, 1993, Berkeley, CA, USA), Elsevier, Amsterdam, 1993, 89-105.
27. Jackson W.B., Jucker J.V., An empirical study of travel time variability and travel choice behavior, TRANSPORT SCI., 1982, 16, 460-475.
28. Brastow W.C., Jucker J.V., Use of a mean variance criterion for traffic assignment, unpublished manuscript, Department of Industrial Engineering, Stanford University, 1977
29. Szeto W.Y., Solayappan M., Jiang Y., Reliability-based transit assignment for congested stochastic transit networks, COMPUT-AIDED CIV INF., 2011, 26, 311-326.
30. Chen A., Zhou Z., The α -reliable mean-excess traffic equilibrium model with stochastic travel times, TRANSPORT RES B-METH., 2010, 44, 493-513.
31. Franklin J.P., Karlstrom A., Travel time reliability for Stockholm roadways: modeling the mean lateness factor, TRANSPORT RES REC., 2009, 2134, 106–113.
32. Chan K.S., Lam W.H.K., Impact of road pricing on the road network reliability, J EAST ASIA SOC TRANSPORT STUD., 2005, 6, 2060-2075.
33. Ordóñez F., Stier-Moses N.E., Wardrop equilibria with risk-averse users, TRANSPORT SCI., 2010, 44, 63-86.
34. Lam T., Small K., The value of time and reliability: measurement from a value pricing experiment, TRANSPORT RES E-LOG., 2001, 37, 231–251.
35. Bell M.G.H., Cassir C., Risk-averse user equilibrium traffic assignment: an application of game theory, TRANSPORT RES B-METH., 2002, 36, 671-681.
36. Szeto W.Y., O'Brien L., O'Mahony M., Generalisation of the risk-averse traffic assignment, In: Bell M.G.H., Heydecker B.G., Allsop R.E. (Eds.), Proceedings of 17th International Symposium on Transportation and Traffic Theory (July 23-25, 2007, London, UK), Emerald Group Publishing Limited, 2007, 127-153.
37. Szeto W.Y., O'Brien L., O'Mahony M., Measuring network reliability by considering paradoxes: the multiple network demon approach, TRANSPORT RES REC., 2009, 2090, 42-50.
38. Szeto W.Y., Cooperative game approaches to measuring network reliability considering paradoxes, TRANSPORT RES C-EMER., 2011, 11, 229-241.
39. Szeto W.Y., O'Brien L., O'Mahony M., Risk-averse traffic assignment with elastic demand: NCP formulation and solution method for assessing performance reliability, NETW SPAT ECON., 2006, 6, 313-332.
40. Ordóñez F., Stier-Moses N.E., Robust Wardrop equilibrium, NETW CONTROL OPTIM., 2007, 4465, 247-256.
41. Zhang C., Chen X., Sumalee A., Robust Wardrop's user equilibrium assignment under stochastic demand and supply: expected residual minimization approach, TRANSPORT RES B-METH., 2011, 45, 534-552.
42. Tversky A., Kahneman D., Advances in prospect theory: cumulative representation of uncertainty, J RISK UNCERTAINTY, 1992, 5, 297-323.
43. Kahneman D., Tversky A., Prospect theory: an analysis of decisions under risk, ECONOMETRICA, 1979, 47, 263-291.
44. Thaler R.H., Tversky A., Kahneman D., Schwartz A., The effect of myopia and loss aversion on risk taking: an experimental test, QUART J ECON., 1997, 112, 647-661.

45. Camerer C.F., Ho T.H., Violations of the betweenness axiom and nonlinearity in probability, J RISK UNCERTAINTY., 1994, 8, 167-196.
46. Avineri E., Prashker J.N., Violations of expected utility theory in route-choice stated preferences: certainty effect and inflating of small probabilities, TRANSPORT RES
47. Avineri E., The effect of reference point on stochastic network equilibrium, TRANSPORT SCI., 2006, 40, 409–420.
48. Mirchandani P., Soroush H., Generalized traffic equilibrium with probabilistic travel times and perceptions, TRANSPORT SCI., 1987, 21, 133-152.
49. Shao H., Lam W.H.K., Tam M.L., A reliability-based stochastic traffic assignment model for network with multiple user classes under uncertainty in demand, NETW SPAT ECON., 2006, 6, 173-204.
50. Chen A., Zhou Z., A stochastic α-reliable mean-excess traffic equilibrium model with probabilistic travel times and perception errors, In: Lam W.H.K., Wong S.C., Lo H.K. (Eds.), Proceedings of 18th International Symposium on Transportation and Traffic Theory 2009: Golden Jubilee (July 16-18, 2009), Springer, 2009, 117-145.
51. Chu Y.L., Work departure time analysis using dogit ordered generalized extreme value model, TRANSPORT RES REC., 2009, 2132, 42-49.
52. Jou R.C., Kitamura R., Weng M.C., Chen C.C., Dynamic commuter departure time choice under uncertainty, TRANSPORT RES A-POL., 2008, 42, 774-783.
53. Lemp J.D., Kockelman K.M., Empirical investigation of continuous logit for departure time choice with Bayesian methods, TRANSPORT RES REC., 2010, 2165, 59-68.
54. Lemp J.D., Kockelman K.M., Damien P., The continuous cross-nested logit model: Formulation and application for departure time choice, TRANSPORT RES B-METH., 2010, 44, 646-661.
55. Friesz T.L., Luque F.J., Tobin R.L., Wie B.W., Dynamic network traffic assignment considered as a continuous time optimal control problem, OPER RES., 1989, 37, 893-901.
56. Vickrey W.S., Congestion theory and transport investment, AM ECON REV., 1969, 59, 251-261.
57. Mahmassani H.S., Herman R., Dynamic user equilibrium departure time and route choice on an idealized traffic arterials, TRANSPORT SCI., 1984, 18, 362-384.
58. Merchant D.K., Nemhauser G.L., A model and an algorithm for the dynamic traffic assignment problem, TRANSPORT SCI., 1978, 12, 183-199.
59. Small, K.A., The scheduling of consumer activities: work trips, AM ECON REV., 1982, 72, 467-479.
60. Ran B., Boyce D.E., Modelling Dynamic Transportation Networks: An Intelligent Transportation System Oriented Approach, 2nd rev. ed., Springer, Berlin, 1996.
61. Vythoulkas P.C., Two models for predicting dynamic stochastic equilibria in urban transportation networks, In: Koshi M. (Ed.), Proceedings of 11th International Symposium on Transportation and Traffic Theory (July 18-20, Yokohama, Japan), Elsevier, 1990, 253-272.
62. Boyce D.E., Ran B., Li I.Y., Considering travellers' risk-taking behavior in dynamic traffic assignment, In: Bell M.G.H., Transportation Networks: Recent Methodological Advances, Elsevier, Oxford, 1999.
63. Szeto W.Y., Jiang Y, Sumalee A., A cell-based model for multi-class doubly stochastic dynamic traffic assignment, COMPUT-AIDED CIV INF., 2011, 26, 595-611.
64. Simon H.A., Models of Bounded Rationality (Vol. 3), The MIT Press, Cambridge, 1997.
65. Simon H.A., A behavioral model of rational choice, QUART J ECON., 1955, 69, 99-

118.
66. Simon H.A., Models of Man, Wiley, New York, 1957.
67. Mahmassani H.S., Chang G.L., On boundedly-rational user equilibrium in transportation systems, TRANSPORT SCI., 1987, 21, 89-99.
68. Szeto W.Y., Lo H.K., Dynamic traffic assignment: properties and extensions, TRANSPORTMETRICA, 2006, 2, 31-52.
69. Lou Y., Yin Y., Lawphongpanich S., Robust congestion pricing under boundedly rational user equilibrium, TRANSPORT RES B-METH., 2010, 44, 15-28.
70. Szeto W.Y., Lo H.K., A cell-based simultaneous route and departure time choice model with elastic demand, TRANSPORT RES B-METH., 2004, 38, 593-612.
71. Pel A.J., Bliemer M.C.J., Hoogendoorn S.P., Hybrid route choice modeling in dynamic traffic assignment, TRANSPORT RES REC., 2009, 2091, 100-107.
72. Kuwahara M., Akamatsu T., Decomposition of the reactive assignments with queues for many-to-many origin-destination pattern, TRANSPORT RES B-METH., 1997, 31, l-10.
73. Ben-akiva M., De Palma A., Kaysi, I., Dynamic network models and driver information system, TRANSPORT RES A-POL., 1991, 25, 251-266.
74. Sumalee A., Zhong, R.X., Pan, T.L., Szeto, W.Y., Stochastic cell transmission model (SCTM): a stochastic dynamic traffic model for traffic state surveillance and assignment, TRANSPORT RES B-METH., 2011, 45, 507-533.
75. Lo H.K., Szeto W.Y., A cell-based dynamic traffic assignment model: formulation and properties, MATH COMPUT MODEL., 2002, 35, 849-865.
76. Tong C.O., Wong S.C., A predictive dynamic traffic assignment model in congested capacity-constrained road networks, TRANSPORT RES B-METH., 2000, 34, 625-644.
77. Horowitz J.L., The stability of stochastic equilibrium in a two-link transportation network, TRANSPORT RES B-METH., 1984, 18, 13-28.
78. Cascetta E., A stochastic process approach to the analysis of temporal dynamics in transportation networks, TRANSPORT RES B-METH., 1989, 23, 1-17.
79. Chang G.L., Mahmassani H., Travel time prediction and departure time adjustment behavior dynamics in a congested traffic system, TRANSPORT RES B-METH., 1988, 22, 217-232.
80. Lam W.H.K., Huang H.J., Dynamic user optimal traffic assignment model for many to one travel demand, TRANSPORT RES B-METH., 1995, 29, 243-259.
81. Ran B., Boyce D., A link-based variational inequality formulation of ideal dynamic user-optimal route choice problem, TRANSPORT RES C-EMER., 1996, 4, 1-12.
82. Carey M., Optimal time-varying flows on congested network, OPER RES., 1987, 35, 58-69.
83. Carey M., Srinivasan A., Congested network flows: time-varying demands and start-time policies, EUR J OPER RES., 1988, 36, 227-240.
84. Lo H.K., Szeto W.Y., A cell-based variational inequality formulation of the dynamic user optimal assignment problem, TRANSPORT RES B-METH., 2002, 36, 421-443.
85. Szeto W.Y., The enhanced lagged cell transmission model for dynamic traffic assignment, TRANSPORT RES REC., 2008, 2085, 76-85.
86. Lo H.K., Ran B., Hongola B., Multiclass dynamic traffic assignment model: formulation and computational experiences, TRANSPORT RES REC., 1996, 1537, 74-82.
87. Liu Y.H., Mahmassani H.S., Dynamic aspects of commuter decisions under advanced traveler information systems – modeling framework and experimental results, TRANSPORT RES REC., 1998, 1645, 111-119.

88. Ben-Akiva M., Cuneo D., Hasan M., Jha M., et al., Evaluation of freeway control using a microscopic simulation laboratory, TRANSPORT RES C-EMER., 2003, 11, 29-50.
89. Jiang Y.Q., Wong S.C., Ho H.W., Zhang P., et al., A dynamic traffic assignment model for a continuum transportation system, TRANSPORT RES B-METH., 2011, 45, 343-363.
90. Hughes R.L., A continuum theory for the flow of pedestrians, TRANSPORT RES B-METH., 2002, 36, 507-535.
91. Hoogendoorn S.P., Bovy P.H.L., Dynamic user-optimal assignment in continuous time and space, TRANSPORT RES B-METH., 2004, 38, 571-592.
92. Hoogendoorn S.P., Bovy P.H.L., Pedestrian route-choice and activity scheduling theory and models, TRANSPORT RES B-METH., 2004, 38, 169-190.
93. Xia Y., Wong S.C., Zhang M., Shu C.W., et al., An efficient discontinuous Galerkin method on triangular meshes for a pedestrian flow model, INT J NUMER METH ENG, 2008, 76, 337-350.
94. Huang L., Wong S.C., Zhang M.P., Shu C.W., et al., Revisiting Hughes' dynamic continuum model for pedestrian flow and the development of an efficient solution algorithm. TRANSPORT RES B-METH., 2009, 43, 127-141.
95. Huang L., Xia Y., Wong S.C., Shu C.W., et al., Dynamic continuum model for bi-directional pedestrian flows, P I CIVIL ENG: ENG COMPUT MEC., 2009, 162, 67-75.
96. Xia Y., Wong S.C., Shu C.W., Dynamic continuum pedestrian flow model with memory effect, PHYS REV E, 2009, 79, 066113.
97. Jiang Y.Q., Xiong T., Wong S.C., Shu C.W., et al., A reactive dynamic continuum user equilibrium model for bi-directional pedestrian flows, ACTA MATH SCI, 2009, 29, 1541-1555.
98. Guo R.Y., Huang H.J., Wong S.C., Collection, spillback, and dissipation in pedestrian evacuation: a network-based method, TRANSPORT RES B-METH., 2011, 45, 490-506.
99. Xiong T., Zhang M., Shu C.W., Wong S.C., et al., High-order computational scheme for a dynamic continuum model for bi-directional pedestrian flows, COMPUT-AIDED CIV INF., 2011, 26, 298-310.
100. Nagurney A., Network Economics: A Variational Inequality Approach, Kluwer Academic Publishers, Norwell, Massachusetts, USA, 1993.
101. Chow A.H.F., Properties of system optimal traffic assignment with departure time choice and its solution method. TRANSPORT RES B-METH., 2009, 43, 325-344.
102. Chow A.H.F., Dynamic system optimal traffic assignment - a state-dependent control theoretic approach. TRANSPORTMETRICA, 2009, 5, 85-106.
103. Long J.C., Szeto W.Y., Gao Z.Y., Discretised link travel time models based on cumulative flows: formulations and properties, TRANSPORT RES B-METH., 2011, 45, 232-254.