which is always negative for values of $\Delta \geq 27.02$. Since the reset control system produces crossings with the reset surface with intervals that are lesser than 27.02, then it is not guaranteed that the reset control system be stable in this case. Fig. 5 shows a simulation for this system, comparing closed loop outputs both for the base and the reset control system, for $\Delta_m = 0.1$. Using again Proposition 4.1, it can be concluded that the closed-loop reset system is asymptotically stable if reset intervals $\Delta_k, k = 0, 1, \ldots$, are always greater than $\Delta_m = 27.1$. Fig. 6 shows a simulation for $\Delta_m = 30$, showing that the system is stable in spite that the base system is oscillating. For values of Δ_m closer to the stability limit the response become increasingly oscillating.

V. Conclusion

Stability conditions dependent on the reset times have been developed for reset control systems. As a result, reset control systems stability is determined by using a time-varying discrete time system describing the evolution of the system after the reset instants. In comparison with previous work, the main contribution has been to include restrictions only at the reset instants, and thus results are less conservative and can be applied to reset systems with both stable and unstable base systems. In addition, the time regularization constant has been used to developed a stabilization result for the case in which the base system is stable. As a result, a lower bound of the reset intervals always exists that guarantee stability of the reset system, if no reset action is performed for reset intervals lower than this bound. Several examples have been analyzed in detail, showing in particular how reset times dependent conditions are less conservative that previous reset systems stability results such as the H_β condition.

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Further Results on Exponential Estimates of Markovian Jump Systems With Mode-Dependent Time-Varying Delays

Huijun Gao, Zhongyang Fei, James Lam, and Baozhu Du

***Abstract*—This technical note studies the problem of exponential estimates for Markovian jump systems with mode-dependent interval time-varying delays. A novel Lyapunov–Krasovskii functional (LKF) is constructed with the idea of delay partitioning, and a less conservative exponential estimate criterion is obtained based on the new LKF. Illustrative examples are provided to show the effectiveness of the proposed results.**

***Index Terms*—Delay partitioning, exponential estimates, Markovian jump systems, mode-dependent time delays.**

I. Introduction

Markovian jump systems have been widely used to model abrupt changes in structures, such as random failures of the components, sudden disturbances, variations of the environment, and changes of the subsystem interconnections. Many researchers have been attracted to this field and a lot of problems have been investigated, including H_∞ control, H_∞ filtering, output control, etc [2], [4], [6], [12], [13], [17]–[19].

On the other hand, the existence of time delays is very common in practical systems, such as network based control systems, chemical processes and communication systems. It is also well known that time

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delays may lead to oscillation, divergence and even instability [9], and a lot of work has been conducted to the systems with time delays [8], [10]. Meanwhile, considerable amount of efforts have been paid to the analysis and synthesis problems for Markovian jump systems with time delays [1], [3], [11], [15].

As one of the most important and fundamental issues to be addressed, the stability characteristics of Markovian jump systems with time delays has been studied by many researchers. A delay-independent exponential stability criterion was proposed in [16]. Since the criterion allows the time delay to be arbitrarily large, it tends to be conservative in general. Special interests have been given to obtain delay-dependent criteria for Markovian systems with time delays. In [20], a simple exponential stability criterion was obtained for Markovian jump system with mode-dependent delays, however, it will turn to be inapplicable when the delays change rapidly. Furthermore, though the decay rate can be computed, it is a fixed value that one cannot adjust to deduce if a larger decay rate is possible. In [14], the authors proposed a new exponential stability criterion by choosing a new Lyapunov-Krasovskii functional and introducing some slack variables to reduce the conservatism caused by bounding techniques. However, these delay-dependent results assume the time delay is constant or varying between zero and an upper bound, while in practice, the lower bound may not be restricted to be zero.

In this technical note, we focus on the exponential stability conditions for Markovian jump systems with mode-dependent interval time-varying delays. To this end, an appropriate Lyapunov–Krasovskii functional (LKF) is established with the idea of partitioning the lower bound, which is inspired by the ideas presented in [5], [7]. Then a less conservative delay-range-dependent exponential stability criterion is derived based on the new LKF, which is efficient in treating both fast and slow time-varying delays. Finally, numerical examples are given to demonstrate the advantages of our method.

Notation: The notation used in this technical note is standard. $\mathbb{R}^n$ represents the n-dimensional Euclidean space. For real symmetric matrices X and Y, the notation $X > Y$ means that the matrix $X - Y$ is positive definite. The superscripts "T" and "-1" represent matrix transposition and matrix inverse respectively; $0_{m,n}$ denotes zero matrix with $m \times n$ dimensions; an asterisk $*$ represents a term that is induced by symmetry. $\|\cdot\|$ stands for the Euclidean norm for matrices and $|\cdot|$ denotes the spectral norm for vectors. $|\psi|_{h_2} = \sup_{-h_2 \le s \le 0} |\psi(s)|$ for the family of any possible continuous ψ. $E\{\cdot\}$ stands for the expectation operator with respect to some probability measure. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated.

II. Problem Formulation

Consider the following linear state-delay systems with Markovian jumping parameters:

$$\dot{x}(t) = A(r(t))x(t) + A_d(r(t))x(t - \tau(r(t), t)) \tag{1}$$

$$x(t) = \psi(t), \quad t \in [-h_2, 0], \quad r(0) = r_0 \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $A(r(t))$ and $A_d(r(t))$ are matrix functions of the random jumping process $\{r(t)\}$, $r(t)$ is a finite state Markov jump process representing the system mode, and takes discrete values in a given finite set $S = \{1, 2, \ldots, N\}$. The transition probability matrix $\Pi = [\pi_{ij}]$ is denoted as follows:

$$\Pr\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta) & i \ne j \\ 1 + \pi_{ii}\Delta + o(\Delta) & i = j \end{cases} \tag{3}$$

where $\Delta > 0$, and $\pi_{ij} \ge 0$, for $i \ne j$, is the transition rate from mode i at time t to mode j at time $t + \Delta$ and

$$\sum_{j=1, j\ne i}^{s} \pi_{ij} = -\pi_{ii} \tag{4}$$

for each mode $i \in S$, $o(\Delta)/\Delta \to 0$. $\tau_i(t)$ denotes the mode dependent time-varying state delay in the system, and satisfies the following condition:

$$0 < h_{1i} \le \tau_i(t) \le h_{2i} < \infty \tag{5}$$

$$\dot{\tau}_i(t) \le \mu_i \tag{6}$$

where h_{1i}, h_{2i} and μ_i are constant for $i \in S$. $\psi(t) \in [-h_2, 0]$ and $r_0 \in S$ are initial conditions of continuous state and the mode, respectively, with $h_1 = \min\{h_{1i}, i \in S\}$ and $h_2 = \max\{h_{2i}, i \in S\}$. $A(r_t)$, $A_d(r_t)$ are matrix functions of the random jumping process $\{r_t\}$ and represent the nominal systems for each $r_t \in S$. For notation simplicity, when the system operates in the i-th mode $(r(t) = i)$, $A(r(t))$ and $A_d(r(t))$ are denoted as A_i and A_{di}, respectively.

Definition 1: [14] The Markovian jump system in (1) is said to be mean square exponentially stable if, for any finite $\psi(t) \in \mathbb{R}^n$ defined on $[-h_2, 0]$ and initial mode $r_0 \in S$, there exist positive constants σ and λ, such that the following condition is satisfied:

$$E|x(t)|^2 \le \sigma e^{-\lambda t} |\psi|_{h_2}^2 \tag{7}$$

where σ and λ are called the decay coefficient and decay rate, respectively.

III. Main Results

In this section, we present our new exponential stability criterion for Markovian jump system with mode-dependent time-varying delays.

Theorem 1: Given a decay rate λ and an integer $m \ge 1$, for time-varying delay $\tau_i(t)$, the Markovian jump system in (1) is mean square exponentially stable, if there exist matrices $P_i > 0, Q_{1i} \ge 0, Q_{2i} \ge 0, Q_{3i} \ge 0, Q_1 \ge 0, Q_2 \ge 0, R_1 > 0, R_2 > 0$, such that the following LMIs hold:

$$\Theta_{1i} = \Xi_i - \frac{1}{h_{12}} W_{R_{2\_1}}^T R_2 W_{R_{2\_1}} < 0 \tag{8}$$

$$\Theta_{2i} = \Xi_i - \frac{1}{h_{12}} W_{R_{2\_2}}^T R_2 W_{R_{2\_2}} < 0 \tag{9}$$

$$e^{\frac{\lambda h_1}{m}} \sum_{j=1}^{N} \pi_{ij} Q_{1j} \le Q_1 \tag{10}$$

$$e^{\lambda h_{12}} \left(\sum_{j=1, j\ne i}^{N} \pi_{ij} Q_{2j} + \sum_{j=1}^{N} \pi_{ij} Q_{3j} \right) \le Q_2 \tag{11}$$

for $i = 1, 2, \ldots, N$, where

$$\begin{aligned}
\Xi_i = {} & W_{P_1}^T \left(\lambda P_i + \sum_{j=1}^{s} \pi_{ij} P_j + \frac{e^{\lambda h_2} - e^{\lambda h_1}}{\lambda} Q_2 \right) W_{P_1} \\
& + sym\left(W_{P_1}^T P_i W_{P_{2i}} \right) + e^{\frac{\lambda h_1}{m}} W_{Q_{1_1}}^T Q_{1i} W_{Q_{1_1}} \\
& + \frac{e^{\frac{\lambda h_1}{m}} - 1}{\lambda} W_{Q_{1_1}}^T Q_1 W_{Q_{1_1}} - W_{Q_{1_2}}^T Q_{1i} W_{Q_{1_2}} \\
& + e^{\lambda h_{12}} W_{Q_{2_1}}^T (Q_{2i} + Q_{3i}) W_{Q_{2_1}} \\
& - W_{Q_{2_2}}^T (Q_{2i} - \mu_i e^{\lambda h_{12}} Q_{2i}) W_{Q_{2_2}} \\
& + W_{R_{1_i}}^T \left(\frac{e^{\frac{\lambda h_1}{m}} - 1}{\lambda} R_1 + \frac{e^{\lambda h_2} - e^{\lambda h_1}}{\lambda} R_2 \right) W_{R_{1_i}}
\end{aligned}$$

$$\begin{aligned}
&- W_{Q_3}^T Q_{3i} W_{Q_3} - \frac{m}{h_1} W_{R_1}^T R_1 W_{R_1} \\
&- \frac{1}{h_{12}} W_{R_{2_1}}^T R_2 W_{R_{2_1}} - \frac{1}{h_{12}} W_{R_{2_2}}^T R_2 W_{R_{2_2}} \\
W_{P_1} &= \left[I_n, 0_{n,(m+2)n}\right], \ W_{P_{2_i}} = [A_i, 0_{n,mn}, A_{di}, 0_n] \\
W_{Q_{1_1}} &= [I_{mn}, 0_{mn,3n}], \ W_{Q_{1_2}} = [0_{mn,n}, I_{mn}, 0_{mn,2n}] \\
W_{Q_{2_1}} &= [0_{n,mn}, I_n, 0_{n,2n}], \ W_{Q_{2_2}} = \left[0_{n,(m+1)n}, I_n, 0_n\right] \\
W_{Q_3} &= \left[0_{n,(m+2)n}, I_n\right], \ W_{R_{1_i}} = [A_i, 0_{n,mn}, A_{di}, 0_n] \\
W_{R_1} &= \left[I_n, -I_n, 0_{n,(m+1)n}\right], \ h_{12} = h_2 - h_1. \\
W_{R_{2_1}} &= [0_{n,mn}, I_n, -I_n, 0_n], \ W_{R_{2_2}} = \left[0_{n,(m+1)n}, I_n, -I_n\right].
\end{aligned}$$

Proof: First in order to cast our model into the framework of the Markov processes, we define a new process $\{(x_t, r_t), t \geq 0\}$ by

$$x_t(s) = x(t+s), \quad t - h_2 \leq s \leq t.$$

Now, take the stochastic Lyapunov–Krasovskii functional candidate as

$$V(x_t, i, t) = e^{\lambda t} V_1(x_t, i, t) + \sum_{j=2}^{4} V_j(x_t, i, t) \tag{12}$$

where

$$\begin{aligned}
V_1(x_t, i, t) &= x^T(t) P_i x(t) \\
V_2(x_t, i, t) &= \int_{t-\frac{h_1}{m}}^{t} e^{\lambda\left(s+\frac{h_1}{m}\right)} \Upsilon^T(s) Q_{1i} \Upsilon(s) ds \\
&+ \int_{t-\tau_i(t)}^{t-h_1} e^{\lambda(s+h_2)} x^T(s) Q_{2i} x(s) ds \\
&+ \int_{t-h_2}^{t-h_1} e^{\lambda(s+h_2)} x^T(s) Q_{3i} x(s) ds \\
V_3(x_t, i, t) &= \int_{-\frac{h_1}{m}}^{0} \int_{t+\theta}^{t} e^{\lambda(s-\theta)} \dot{x}^T(s) R_1 \dot{x}(s) ds d\theta \\
&+ \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} e^{\lambda(s-\theta)} \dot{x}^T(s) R_2 \dot{x}(s) ds d\theta \\
V_4(x_t, i, t) &= \int_{-\frac{h_1}{m}}^{0} \int_{t+\theta}^{t} e^{\lambda(s-\theta)} \Upsilon^T(s) Q_1 \Upsilon(s) ds d\theta \\
&+ \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} e^{\lambda(s-\theta)} x^T(s) Q_2 x(s) ds d\theta
\end{aligned}$$

with

$$\Upsilon(s) = \left[x^T(s), x^T\left(s - \frac{1}{m} h_1\right), \cdots, x^T\left(s - \frac{m-1}{m} h_1\right)\right]^T.$$

The weak infinitesimal operator $\mathcal{F}$ is defined as [19]

$$\mathcal{F}V(x_t, r_t, t) = \lim_{\Delta \to 0_+} \frac{1}{\Delta} \left[E\left\{V\left(x(t+\Delta), r(t+\Delta), t+\Delta\right)\right\} - V(x_t, r_t, t)\right].$$

Then we have

$$\begin{aligned}
\mathcal{F}V_1 &= x^T(t) \sum_{j=1}^{s} \pi_{ij} P_j x(t) + 2x^T(t) P_i \dot{x}(t) \\
\mathcal{F}V_2 &\leq e^{\lambda\left(t+\frac{h_1}{m}\right)} \Upsilon^T(t) Q_{1i} \Upsilon(t) \\
&- e^{\lambda t} \Upsilon^T\left(t - \frac{h_1}{m}\right) Q_{1i} \Upsilon\left(t - \frac{h_1}{m}\right) \\
&+ \int_{t-\frac{h_1}{m}}^{t} e^{\lambda\left(s+\frac{h_1}{m}\right)} \Upsilon^T(s) \left(\sum_{j=1}^{N} \pi_{ij} Q_{1j}\right) \Upsilon(s) ds \\
&+ e^{\lambda(t+h_2-h_1)} x^T(t-h_1) Q_{2i} x(t-h_1) \\
&- e^{\lambda t} x^T\left(t - \tau_i(t)\right) Q_{2i} x\left(t - \tau_i(t)\right) \\
&+ \mu_i e^{\lambda(t+h_{12})} x^T\left(t - \tau_i(t)\right) Q_{2i} x\left(t - \tau_i(t)\right) \\
&+ \sum_{j=1}^{N} \pi_{ij} \int_{t-\tau_j(t)}^{t-h_1} e^{\lambda(s+h_2)} x^T(s) Q_{2j} x(s) ds \\
&+ e^{\lambda(t+h_{12})} x^T(t-h_1) Q_{3i} x(t-h_1) \\
&- e^{\lambda t} x^T(t-h_2) Q_{3i} x(t-h_2) \\
&+ \int_{t-h_2}^{t-h_1} e^{\lambda(s+h_2)} x^T(s) \left(\sum_{j=1}^{N} \pi_{ij} Q_{3j}\right) x(s) ds \\
\mathcal{F}V_3 &= e^{\lambda t} \dot{x}^T(t) R_1 \dot{x}(t) \frac{e^{\frac{\lambda h_1}{m}} - 1}{\lambda} \\
&- e^{\lambda t} \int_{t-\frac{h_1}{m}}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds \\
&+ e^{\lambda t} \dot{x}^T(t) R_2 \dot{x}(t) \frac{e^{\lambda h_2} - e^{\lambda h_1}}{\lambda} \\
&- e^{\lambda t} \int_{t-h_2}^{t-h_1} \dot{x}^T(s) R_2 \dot{x}(s) ds \\
\mathcal{F}V_4 &= e^{\lambda t} \Upsilon^T(t) Q_1 \Upsilon(t) \frac{e^{\frac{\lambda h_1}{m}} - 1}{\lambda} \\
&- e^{\lambda t} \int_{t-\frac{h_1}{m}}^{t} \Upsilon^T(s) Q_1 \Upsilon(s) ds \\
&+ e^{\lambda t} x^T(t) Q_2 x(t) \frac{e^{\lambda h_2} - e^{\lambda h_1}}{\lambda} \\
&- e^{\lambda t} \int_{t-h_2}^{t-h_1} x^T(s) Q_2 x(s) ds.
\end{aligned} \tag{13}$$

Noticing $\pi_{ij} \geq 0$ for $j \neq i$ and $\pi_{ii} \leq 0$, then we have

$$\begin{aligned}
&\sum_{j=1}^{N} \pi_{ij} \int_{t-\tau_j(t)}^{t-h_1} e^{\lambda(s+h_2)} x^T(s) Q_{2j} x(s) ds \\
&\leq \int_{t-h_2}^{t-h_1} e^{\lambda(s+h_2)} x^T(s) \left(\sum_{j=1, j\neq i}^{N} \pi_{ij} Q_{2j}\right) x(s) ds.
\end{aligned} \tag{14}$$

Suppose $\beta_i = (\tau_i(t) - h_1)/(h_2 - h_1)$, we have the following equations:

$$\begin{aligned}
&- \int_{t-\tau_i(t)}^{t-h_1} \dot{x}^T(s) R_2 \dot{x}(s) ds \\
&\leq - \frac{1-\beta_i}{h_{12}} \int_{t-\tau_i(t)}^{t-h_1} \left(\tau_i(t) - h_1\right) \dot{x}^T(s) R_2 \dot{x}(s) ds \\
&- \frac{1}{h_{12}} \int_{t-\tau_i(t)}^{t-h_1} \left(\tau_i(t) - h_1\right) \dot{x}^T(s) R_2 \dot{x}(s) ds
\end{aligned}$$

$$
\begin{aligned}
&-\int_{t-h_2}^{t-\tau_i(t)} \dot{x}^T(s)R_2\dot{x}(s)ds \\
&\quad\leq -\frac{1}{h_{12}}\int_{t-h_2}^{t-\tau_i(t)} (h_2-\tau_i(t))\,\dot{x}^T(s)R_2\dot{x}(s)ds \\
&\quad\quad -\frac{\beta_i}{h_{12}}\int_{t-h_2}^{t-\tau_i(t)} (h_2-\tau_i(t))\,\dot{x}^T(s)R_2\dot{x}(s)ds.
\end{aligned}
$$

By using Jensen's inequality, we have

$$
\begin{aligned}
&-\int_{t-\frac{h_1}{m}}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds \\
&\quad\leq -\frac{m}{h_1}\left[x(t)-x\left(t-\frac{h_1}{m}\right)\right]^T \\
&\quad\quad\times R_1\left[x(t)-x\left(t-\frac{h_1}{m}\right)\right] \qquad (15)\\
&-\int_{t-\tau_i(t)}^{t-h_1} (\tau_i(t)-h_1)\,\dot{x}^T(s)R_2\dot{x}(s)ds \\
&\quad\leq -\left[x(t-h_1)-x(t-\tau_i(t))\right]^T \\
&\quad\quad\times R_2\left[x(t-h_1)-x(t-\tau_i(t))\right] \qquad (16)\\
&-\int_{t-h_2}^{t-\tau_i(t)} (h_2-\tau_i(t))\,\dot{x}^T(s)R_2\dot{x}(s)ds \\
&\quad\leq -\left[x(t-\tau_i(t))-x(t-h_2)\right]^T \\
&\quad\quad\times R_2\left[x(t-\tau_i(t))-x(t-h_2)\right]. \qquad (17)
\end{aligned}
$$

Based on (13)–(17), we obtain

$$
\begin{aligned}
\mathcal{F}V \leq\; & e^{\lambda t}\zeta^T(t)\left[(1-\beta_i)\Theta_{1i}+\beta_i\Theta_{2i}\right]\zeta(t) \\
&+\int_{t-\frac{h_1}{m}}^{t} e^{\lambda\left(s+\frac{h_1}{m}\right)}\Upsilon^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{1j}\right)\Upsilon(s)ds \\
&-e^{\lambda t}\int_{t-\frac{h_1}{m}}^{t} \Upsilon^T(s)Q_1\Upsilon(s)ds \\
&+\int_{t-h_2}^{t-h_1} e^{\lambda(s+h_2)}x^T(s) \\
&\quad\times\left(\sum_{j=1,j\neq i}^{N}\pi_{ij}Q_{2j}+\sum_{j=1}^{N}\pi_{ij}Q_{3j}\right)x(s)ds \\
&-e^{\lambda t}\int_{t-h_2}^{t-h_1} x^T(s)Q_2x(s)ds \qquad (18)
\end{aligned}
$$

where

$$
\zeta(t)=\left[\Upsilon^T(t),x^T(t-h_1),x^T(t-\tau_i(t)),x^T(t-h_2)\right]^T.
$$

If $\sum_{j=1}^{N}\pi_{ij}Q_{1j}\geq 0$ and $\sum_{j=1,j\neq i}^{N}\pi_{ij}Q_{2j}+\sum_{j=1}^{N}\pi_{ij}Q_{3j}\geq 0$, then we have

$$
\begin{aligned}
&\int_{t-\frac{h_1}{m}}^{t} e^{\lambda\left(s+\frac{h_1}{m}\right)}\Upsilon^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{1j}\right)\Upsilon(s)ds \\
&\quad\leq\int_{t-\frac{h_1}{m}}^{t} e^{\lambda\left(t+\frac{h_1}{m}\right)}\Upsilon^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{1j}\right)\Upsilon(s)ds \\
&\int_{t-h_2}^{t-h_1} e^{\lambda(s+h_2)}x^T(s)\left(\sum_{j=1,j\neq i}^{N}\pi_{ij}Q_{2j}+\sum_{j=1}^{N}\pi_{ij}Q_{3j}\right)x(s)ds \\
&\quad\leq\int_{t-h_2}^{t-h_1} e^{\lambda(t+h_{12})}x^T(s) \\
&\quad\quad\times\left(\sum_{j=1,j\neq i}^{N}\pi_{ij}Q_{2j}+\sum_{j=1}^{N}\pi_{ij}Q_{3j}\right)x(s)ds.
\end{aligned}
$$

As $0\leq\beta_i\leq 1$, it is obvious that $\mathcal{F}V<0$ if (8), (9), (10) and (11) hold.

If $\sum_{j=1}^{N}\pi_{ij}Q_{1j}<0$ and $\sum_{j=1,j\neq i}^{N}\pi_{ij}Q_{2j}+\sum_{j=1}^{N}\pi_{ij}Q_{3j}<0$, then the following inequality is easy to get from (18):

$$
\mathcal{F}V\leq e^{\lambda t}\zeta^T(t)\left[(1-\beta_i)\Theta_{1i}+\beta_i\Theta_{2i}\right]\zeta(t).
$$

For $0\leq\beta_i\leq 1$, $\mathcal{F}V<0$ if (8), (9) hold. Meanwhile, (10) and (11) also hold in this case, since

$$
\begin{aligned}
e^{\frac{\lambda h_1}{m}}\sum_{j=1}^{N}\pi_{ij}Q_{1j}&<0<Q_1 \\
e^{\lambda h_{12}}\left(\sum_{j=1,j\neq i}^{N}\pi_{ij}Q_{2j}+\sum_{j=1}^{N}\pi_{ij}Q_{3j}\right)&<0<Q_2.
\end{aligned}
$$

With the same method, we can also deal with the other two cases $\sum_{j=1}^{N}\pi_{ij}Q_{1j}<0$ with $\sum_{j=1,j\neq i}^{N}\pi_{ij}Q_{2j}+\sum_{j=1}^{N}\pi_{ij}Q_{3j}>0$, and $\sum_{j=1}^{N}\pi_{ij}Q_{1j}>0$ with $\sum_{j=1,j\neq i}^{N}\pi_{ij}Q_{2j}+\sum_{j=1}^{N}\pi_{ij}Q_{3j}<0$, and find that if (8), (9), (10) and (11) are satisfied, $\mathcal{F}V<0$ can be guaranteed.

For any $t>0$, we have

$$
\begin{aligned}
|x(t)|&=\left|x(0)+\int_{0}^{t}\left[A_ix(s)+A_{di}x(s-\tau_i(t))\right]ds\right| \\
&\leq|x(0)|+\int_{0}^{t}d_1|x(s)|ds+\int_{0}^{t}d_2|x(s-\tau_i(t))|ds
\end{aligned}
$$

where $d_1=\max_{i\in S}\{\|A_i\|\}$, $d_2=\max_{i\in S}\{\|A_{di}\|\}$. When $0\leq t\leq h_2$, we have

$$
|x(t)|\leq[d_2h_2+1]|\psi|_{h_2}+\int_{0}^{t}d_1|x(s)|ds. \qquad (19)
$$

By the Gronwall–Bellman Lemma, we have the following inequality:

$$
|x(t)|\leq[d_2h_2+1]e^{d_1h_2}|\psi|_{h_2}=d|\psi|_{h_2}
$$

where $d=(d_2h_2+1)e^{d_1h_2}$.

For any $-h_2\leq t-\tau_i(t)\leq h_2$, the following inequality is true:

$$
|x(t-\tau_i(t))|\leq\max\{1,d\}|\psi|_{h_2}=d|\psi|_{h_2}.
$$

When $t>h_2$, by Dynkin's formula [20], we have

$$
\begin{aligned}
EV(x_t,i,t)&=EV(x_{h_2},i,h_2)+E\int_{0}^{t-h_2}\mathcal{F}V(x_s,i,s)ds \\
&\leq\sum_{\varrho=1}^{5}\alpha_\varrho|\psi|_{h_2}^2
\end{aligned}
$$

where

$$\alpha_1 = d^2 e^{\lambda h_2} \max_{i\in S}\{\|P_i\|\}$$
$$\alpha_2 = \max_{i\in S}\{\|Q_{1i}\|\} m d^2 e^{\lambda h_2}\frac{e^{\frac{\lambda h_1}{m}}-1}{\lambda}$$
$$\alpha_3 = \left(\max_{i\in S}\{\|Q_{2i}\|\}+\max_{i\in S}\{\|Q_{3i}\|\}\right) d^2 e^{\lambda h_2}\frac{e^{\lambda h_{12}}-1}{\lambda}$$
$$\alpha_4 = (d_1 d + d_2 d)^2 \frac{e^{\lambda h_2}}{\lambda^2}\left[\|R_1\|\left(e^{\frac{\lambda h_1}{m}}-\frac{\lambda h_1}{m}-1\right) + \|R_2\|(e^{\lambda h_2}-e^{\lambda h_1}-\lambda h_{12})\right]$$
$$\alpha_5 = d^2\frac{e^{\lambda h_2}}{\lambda^2}\left[m\|Q_1\|\left(e^{\frac{\lambda h_1}{m}}-\frac{\lambda h_1}{m}-1\right) + \|Q_2\|(e^{\lambda h_2}-e^{\lambda h_1}-\lambda h_{12})\right]$$
$$\alpha_6 = \max_{i\in S}\left\{\left\|P_i^{-1}\right\|\right\}.$$

On the other hand

$$EV(x_t,i,t) \geq \frac{1}{\max_{i\in S}\left\{\left\|P_i^{-1}\right\|\right\}} e^{\lambda t} E\,|x(t)|^2. \tag{20}$$

So, when $t \geq h_2$

$$E\,|x(t)|^2 \leq \alpha_6 \sum_{\varrho=1}^{5} \alpha_\varrho e^{-\lambda t}|\psi|_{h_2}^2 \equiv \alpha(h_1,h_2)e^{-\lambda t}|\psi|_{h_2}^2. \tag{21}$$

For the case $h_1 \leq t < h_2$, $0 < t < h_1$, with the same method, it is easy to find that

$$E\,|x(t)|^2 < \alpha(h_1,h_2)e^{-\lambda t}|\psi|_{h_2}^2. \tag{22}$$

According to (21) and (22), when $t > 0$, we have

$$E\,|x(t)|^2 \leq \alpha(h_1,h_2)e^{-\lambda t}|\psi|_{h_2}^2.$$

So from Definition 1, we know that the Markovian jump system in (1) is exponentially stable with the decay rate λ. Then the proof is completed. □

Remark 1: The new criterion is based on the idea of delay partitioning. Here, we treat with the interval time-varying delay as two parts: the constant delay part h_1 and the time-varying part $h_2 - h_1$, then partition the constant part. The proposed result is much less conservative, and the conservatism will be reduced with the partitioning number m increases, which will be well illustrated with some numerical examples.

From the proof of Theorem 1, we can easily have the following corollary, which dedicates the way to calculate decay coefficient σ.

Corollary 1: For a given decay rate λ and an integer $m \geq 1$, if there exist $P_i > 0$, $Q_{1i} \geq 0$, $Q_{2i} \geq 0$, $Q_{3i} \geq 0$, $Q_1 \geq 0$, $Q_2 \geq 0$, $R_1 > 0$, $R_2 > 0$, such that (8), (9), (10) and (11) hold for any $i \in S$, then an estimate of the decay coefficient for Markovian jump system with mode dependent time-varying delays is given by $\alpha(h_1,h_2)$, which is already defined in (21).

With the case that $h_1 = h_2 = \tau > 0$, the mode-dependent time delay $\tau_i(t)$ becomes a constant one. Then we can have the following proposition.

Proposition 1: Given a decay rate λ and an integer $m \geq 1$, for constant time delay τ, the Markovian jump system in (1) is mean square exponentially stable, if there exist matrices $P_i > 0$, $Q_i \geq 0$, $Q \geq 0$, $R > 0$ such that the following LMIs hold:

$$\begin{aligned}\bar{\Theta}_i = {} & \bar{W}_{P_1}^T\left(\lambda P_i + \sum_{j=1}^{s}\pi_{ij}P_j\right)\bar{W}_{P_1} + sym\left(\bar{W}_{P_1}^T P_i \bar{W}_{P_{2i}}\right) - \bar{W}_{Q_2}^T Q_i \bar{W}_{Q_2} \\ & + \bar{W}_{Q_1}^T\left(e^{\lambda\frac{\tau}{m}}Q_i + \frac{e^{\frac{\lambda\tau}{m}}-1}{\lambda}Q\right)\bar{W}_{Q_1} \\ & + \frac{e^{\frac{\lambda\tau}{m}}-1}{\lambda}\bar{W}_{R_{1i}}^T R \bar{W}_{R_{1i}} - \frac{m}{\tau}\bar{W}_{R_1}^T R \bar{W}_{R_1} \\ & < 0\end{aligned} \tag{23}$$

$$e^{\frac{\lambda\tau}{m}}\sum_{j=1}^{N}\pi_{ij}Q_j \leq Q \tag{24}$$

for $i = 1, 2, \ldots, N$, where

$$\begin{aligned}\bar{W}_{P_1} &= [I_n, 0_{n,mn}], & \bar{W}_{P_{2i}} &= \left[A_i, 0_{n,(m-1)n}, A_{di}\right] \\ \bar{W}_{Q_1} &= [I_{mn}, 0_{mn,n}], & \bar{W}_{R_{1i}} &= \left[A_i, 0_{n,(m-1)n}, A_{di}\right] \\ \bar{W}_{Q_2} &= [0_{mn,n}, I_{mn}], & \bar{W}_{R_1} &= \left[I_n, -I_n, 0_{n,(m-1)n}\right].\end{aligned}$$

Proof: Choose a Lyapunov-Krasovskii functional candidate as follows:

$$V(x_t,i,t) = e^{\lambda t}V_1(x_t,i) + V_2(x_t,i,t) + V_3(x_t,i,t) \tag{25}$$

where

$$V_1(x_t,i) = x^T(t)P_i x(t)$$
$$V_2(x_t,i,t) = \int_{t-\frac{\tau}{m}}^{t} e^{\lambda\left(s+\frac{\tau}{m}\right)}\bar{\Upsilon}^T(s)Q_i\bar{\Upsilon}(s)ds$$
$$V_3(x_t,i,t) = \int_{-\frac{\tau}{m}}^{0}\int_{t+\theta}^{t} e^{\lambda(s-\theta)}\dot{x}^T(s)R\dot{x}(s)dsd\theta + \int_{-\frac{\tau}{m}}^{0}\int_{t+\theta}^{t} e^{\lambda(s-\theta)}\bar{\Upsilon}^T(s)Q\bar{\Upsilon}(s)dsd\theta$$

with

$$\bar{\Upsilon}(s) = \left[x^T(s)\;\; x^T\left(s-\frac{\tau}{m}\right)\;\; \cdots\;\; x^T\left(s-\frac{m-1}{m}\tau\right)\right]^T.$$

Then with the same method presented in proving Theorem 1, we can obtain the result in Proposition 1. □

Remark 2: It is worth mentioning that even with the case $m = 1$, our method still outperforms that in [14]. The reduced conservatism of Proposition 1 benefits from $V_2(x_t,i,t)$, which is selected to be mode dependent in our technical note. Moreover, the proposed proposition requires less computational complexity than the method in [14] in terms of the number of variables. There are $(2.5s+1)n^2 + (0.5s+1)n$ variables needed to solve Theorem 1 in [14], compared with $(s+1)n^2 + (s+1)n$ needed by Proposition 1. As s and n are all positive numbers, the advantage of our criterion in computational complexity is obvious, especially when s and n are large.

Corollary 2: For a given decay rate λ and an integer $m \geq 1$, if there exist $P_i > 0$, $Q_i \geq 0$, $Q \geq 0$, $R > 0$ such that (23) and (24) hold for any $i \in S$, then the estimate of decay coefficient for Markovian jump system with constant time delay is given by

$$\bar{\alpha}(\tau) = \bar{\alpha}_5(\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4) \tag{26}$$

where

$$\bar{\alpha}_1 = \bar{d}^2 e^{\lambda\tau}\max_{i\in S}\{\|P_i\|\}$$
$$\bar{\alpha}_2 = \max_{i\in S}\{\|Q_i\|\} m\bar{d}^2 e^{\lambda\tau}\frac{e^{\frac{\lambda\tau}{m}}-1}{\lambda}$$

TABLE I
MAXIMUM DELAY BOUND τ VIA DIFFERENT METHODS

π_{11}	-0.2	-0.5	-1
Shu *et al.* [14]	0.352	0.349	0.346
Proposition 1, $m=1$	0.374	0.375	0.377
Proposition 1, $m=2$	0.413	0.414	0.417
Proposition 1, $m=3$	0.420	0.421	0.424
Proposition 1, $m=5$	0.423	0.425	0.428

TABLE II
MAXIMUM DECAY RATE λ VIA DIFFERENT METHODS

π_{11}	-0.2	-0.5	-1
Shu *et al.* [14]	0.760	0.703	0.656
Proposition 1, $m=1$	0.931	0.941	0.964
Proposition 1, $m=2$	1.342	1.351	1.388
Proposition 1, $m=3$	1.413	1.425	1.462
Proposition 1, $m=5$	1.451	1.464	1.501

TABLE III
MAXIMUM DELAY BOUND VIA DIFFERENT λ

Theorem 1	$\lambda=0.3$	$\lambda=0.6$	$\lambda=0.9$
$m=1$	0.589	0.524	–
$m=2$	0.643	0.556	0.510
$m=3$	0.650	0.561	0.516
$m=5$	0.654	0.564	0.520

$$\bar{\alpha}_3 = \|R\|(d_1\bar{d}+d_2\bar{d})^2\frac{e^{\lambda\tau}}{\lambda^2}\left(e^{\frac{\lambda\tau}{m}}-\frac{\lambda\tau}{m}-1\right)$$
$$\bar{\alpha}_4 = m\|Q\|\bar{d}^2\frac{e^{\lambda\tau}}{\lambda^2}\left(e^{\frac{\lambda\tau}{m}}-\frac{\lambda\tau}{m}-1\right)$$
$$\bar{\alpha}_5 = \max_{i\in S}\left\{\left\|P_i^{-1}\right\|\right\}, \quad \bar{d}=(d_2\tau+1)e^{d_1\tau}.$$

IV. NUMERICAL EXAMPLES

In this section, we will use two numerical examples to illustrate the advantages of the proposed criteria in this technical note.

Example 1: Consider the following Markovian jump system with constant time-delay:

$$A_1=\begin{bmatrix}-2.25 & -1.44\\ -1.59 & -2.93\end{bmatrix}, \quad A_{d1}=\begin{bmatrix}-0.71 & 1.19\\ 0.67 & -3.20\end{bmatrix}$$
$$A_2=\begin{bmatrix}-1.90 & 0.82\\ 0.69 & -0.79\end{bmatrix}, \quad A_{d2}=\begin{bmatrix}-1.52 & -1.60\\ -0.16 & -1.24\end{bmatrix}.$$

For a prescribed decay rate $\lambda=1.2$ and a given π_{11}, we assume $\pi_{22}=-0.3$, the comparison results of time delay τ via different methods are given in Table I. Then fix time delay $\tau=0.4$ to find the maximum decay rate λ between different methods, which is presented in Table II.

From Table I and Table II, we can see that even with the case $m=1$, our results still outperform those in [14].

Example 2: Consider a Markovian jump system in (1) and (2) with two modes and the following parameters:

$$A_1=\begin{bmatrix}-2.3 & 0.8\\ 1.0 & -2.9\end{bmatrix}, \quad A_{d1}=\begin{bmatrix}0.8 & 1.2\\ 0.7 & -3.5\end{bmatrix}$$
$$A_2=\begin{bmatrix}-1.9 & 0.2\\ 0.6 & -0.8\end{bmatrix}, \quad A_{d2}=\begin{bmatrix}1.3 & -2.6\\ 0.5 & -1.4\end{bmatrix}.$$

The time delays satisfy the following conditions $\dot{\tau}_1(t)\leq 0.2$, $\dot{\tau}_2(t)\leq 0.15$, and the transition probability matrix is $\Pi=\begin{bmatrix}-0.1 & 0.1\\ 0.5 & -0.5\end{bmatrix}$.

First, for given $h_1=0.5$, we record the upper bounds h_2 of the time delays via different decay rate λ in Table III, which satisfy (8), (9), (10) and (11). Then, we record the maximum decay rate λ via different upper bounds h_2 in Table IV.

It can be seen from Table III and Table IV that the conservatism will be reduced with m increasing.

TABLE IV
MAXIMUM DECAY RATE λ VIA DIFFERENT UPPER DELAY BOUNDS h_2

Theorem 1	$h_2=0.6$	$h_2=0.7$
$m=1$	0.277	0.103
$m=2$	0.417	0.180
$m=3$	0.442	0.193
$m=5$	0.455	0.200

V. CONCLUSION

In this technical note, new criteria for exponential stability of Markovian jump system with mode-dependent time delays have been established in terms of LMIs. With the idea of partitioning the lower bound of the interval time delays, a new Lyapunov Krasovskii functional has been constructed, and novel exponential stability criteria with guaranteed decay rate is obtained for the system. The method to estimate relevant decay coefficient is also presented. Numerical examples have illustrated the merits of the proposed criteria, which are less conservative than existing result. The method presented in the technical note could also be extended to treat with filter and controller design for Markovian jump systems with mode-dependent time delays, and also could be used to networked control systems with integrated communication delay and multiple package dropouts.

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Reducing the Complexity of the Sum-of-Squares Test for Stability of Delayed Linear Systems

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***Abstract*—This technical note considers the problem of reducing the computational complexity associated with the Sum-of-Squares approach to stability analysis of time-delay systems. Specifically, this technical note considers systems with a large state-space but where delays affect only certain parts of the system. This yields a coefficient matrix of the delayed state with low rank—a common scenario in practice. The technical note uses the general framework of coupled differential-difference equations with delays in feedback channels. This framework includes systems of both the neutral and retarded-type. The approach is based on recent results which introduced a new Lyapunov–Krasovskii structure which was shown to be necessary and sufficient for stability of this class of systems. This technical note shows how exploiting the structure of the new functional can yield dramatic improvements in computational complexity. Numerical examples are given to illustrate this improvement.**

***Index Terms*—Complexity, Lyapunov–Krasovskii functional, semidefinite programming, sum-of-squares, time delay.**

I. Introduction

In this technical note, we consider stability of linear time-delay systems with fixed delays. The existence of a monotonically decreasing quadratic Lyapunov function is necessary and sufficient for stability of these systems [6], [9], [15]. As is customary, we refer to these Lyapunov functions as Lyapunov-Krasovskii functionals as the state-space is infinite dimensional. The problem of finding such a functional is considered computationally intractable. An obvious solution is to use simplified versions of the functional. Naturally, however, stability conditions derived in such a manner will be conservative [6]. A solution

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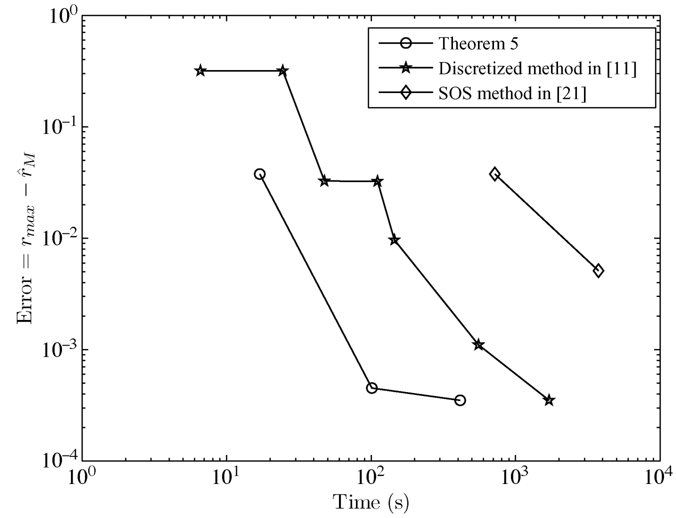

Fig. 1. Log-log plot of accuracy versus computation time: From left, the '∘'-points denote the cases of monomial degree $d = 1, 2, 3$; the '⋆'-points denote the cases with gridding $N = [1, 1], [1, 2], [2, 2], [2, 3], [3, 3], [4, 4], [5, 5]$; the '⋄'-points denote the cases with monomial degree $d = 1, 2$.

to this dilemma was proposed in [4] which used a "discretized" version of the Lyapunov-Krasovskii functional. The product was a series of sufficient conditions which appears to converge to necessity as the level of discretization is increased. The significance of this work is that it gives a quantifiable tradeoff between computational complexity and accuracy of the stability test. See Fig. 1 in the numerical example. In [20] and [21], the problem was approached using polynomials instead of discretized functionals. We refer to this result as the Sum-of-Squares (SOS) method. The advantage of the Sum-of-Squares approach is that it is easily generalized to nonlinear and uncertain systems [17]. It should be pointed out that it is possible to asymptotically approach the analytical limit of stability without the complete quadratic Lyapunov-Krasovskii functional. An interesting method that accomplishes this is the delay partitioning method described in [3]. In all of the above cases, the conditions are expressed using semidefinite programming (SDP) [14], [18]. A problem with both the discretized functional method and the Sum-of-Squares method is that the computational cost increases quickly for large systems with multiple delays.

In most practical systems, although the number of state variables is rather large, there are relatively few delayed elements and these delayed elements enter through low-rank coefficient matrices. Examples include a nuclear reactor model described in [10, Eq. (3.1), Ch. 2]; chemostat models in microbiology described in [10, Eq. (5.4), Ch. 2]; or any controlled system with delayed feedback. However, this feature is not typically leveraged when deriving stability conditions. In this technical note, we reformulate the standard model of time-delayed equations by using coupled differential-difference equations with a single delay in each feedback channel. The idea is that if the dimension of the feedback channel is substantially smaller than the number of states, then this formulation allows us to exploit this low-dimensional structure to potentially reduce the computational cost of stability analysis [7]. In addition, using coupled differential-difference equations allows us to address a larger class of systems that includes time-delay systems of both retarded and neutral type.

Coupled differential-difference equations have been studied for some time. See [2], [22] and [25]. Asymptotic stability analysis based on the input-to-state stability of the difference equations was given in [23]. This result was strengthened to uniform asymptotic stability and extended to the general coupled differential-functional equations in [8], which also considered the possibility of reducing the complexity of the discretized Lyapunov-Krasovskii functional method.