Technical Notes and Correspondence

H_∞ Positive Filtering for Positive Linear Discrete-Time Systems: An Augmentation Approach

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Abstract—**In this note, we address the reduced-order positive filtering problem of positive discrete-time systems under the H_∞ performance. Commonly employed approaches, such as linear transformation and elimination technique, may not be applicable in general due to the positivity constraint of the filter. To cope with the difficulty, we first represent the filtering error system as a singular system by means of the system augmentation approach, which will facilitate the consideration of the positivity constraint. Two necessary and sufficient conditions are obtained in terms of matrix inequalities under which the filtering error system has a prescribed H_∞ performance. Then, a necessary and sufficient condition is proposed for the existence of the desired positive filters, and an iterative linear matrix inequality (LMI) algorithm is presented to compute the filtering matrices, which can be easily checked by standard software. Finally, a numerical example to illustrate the effectiveness of the proposed design procedures is presented.**

Index Terms—**Discrete-time systems, H_∞ filtering, linear matrix inequality, positive filtering, positive systems.**

I. Introduction

Positive systems are dynamic systems with state variables and outputs constrained to be positive (or at least nonnegative) at all times whenever the initial condition and input are nonnegative. Such systems are often encountered and widely used in various areas, for example, biomedicine, pharmacokinetics, ecosystems, economics, industrial engineering involving chemical reactors, heat exchangers, and storage systems [1]. The mathematical theory of positive systems can be traced back to the study of nonnegative matrices proposed by Perron and Frobenius, known as the famous Perron-Frobenius Theorem. For references, we refer readers to [1], [2], and some recent developments can be found in [3], [4].

In view of the widespread applications, it is necessary to investigate the analysis and synthesis problem for positive systems. However, a lot of well-established results for general linear systems cannot be simply applied to positive systems due to the fact that these systems are defined on cones rather than linear spaces. For example, the reachability property, for general linear systems, still holds under similarity transformations, whereas for positive linear systems, this feature may not be true owing to the positivity constraints on system matrices. Therefore, an increasing number of researchers have devoted their time and effort to exploring this kind of systems, and many fundamental results have been reported, see [5]–[8] and references therein. Recent results on the stability and constrained control of positive systems with time delays can be found in [9], [10].

It should be pointed out that, up to now, most existing results on positive systems have dealt with the behavioral analysis, property characterization, and stabilization, whereas the filter/observer problem seems to have received relatively less attention despite its practical significance. Based on the structural decomposition of system matrices, a constructive approach has been established to design positive observers for linear compartmental systems in [11], where the observer is the full-order Luenberger type observer, which should be designed according to the original coordinate system. A Sylvester equation approach has been proposed to study the existence of positive observers of positive linear systems in [12], in which the inverse positivity of a transformation matrix should be firstly guaranteed. It is obvious that both these approaches may result in some restrictions on the structure of positive observers, not to mention the fact that these results are no longer applicable for positive systems of high order and with model uncertainties. Thus, to overcome these difficulties, we may introduce a reduced-order positive H_∞ filter to replace the positive observer. Although the filtering problem has been extensively studied in the control and signal processing communities (see [13]–[15] and references therein), the filtering problem for positive systems still remains to be important and challenging, since it is naturally desirable that the estimator is positive, like the signals to be determined.

In this note, we are concerned with the reduced-order positive H_∞ filter for positive linear discrete-time systems. By means of system augmentation, we first formulate the filtering error system to be a singular system, which assembles the filtering matrices into a parametric matrix to facilitate the analysis and synthesis under the positivity constraint. Then, we propose necessary and sufficient conditions for the filtering error system to have a prescribed H_∞ performance in terms of matrix inequalities. A necessary and sufficient condition is presented to establish the existence of a reduced-order positive H_∞ filter, and an iterative linear matrix inequality algorithm is developed to design the filtering matrix parameters, which can be expressed explicitly.

The rest of this note is organized as follows. Section II presents some notations and preliminaries. Section III is devoted to analysis and synthesis of positive filters for positive linear discrete-time systems. A numerical example is provided in Section IV to demonstrate the benefits of our theoretical results. Finally, we summarize our results in Section V.

II. System Description and Preliminaries

Notation: Let $\mathbb{R}$ be the set of real numbers; $\mathbb{R}^n$ denotes the n-column vectors; $\mathbb{R}^{n\times m}$ is the set of all real matrices of dimension $n \times m$. $\mathbb{R}_+^{n\times m}$ represents the $n \times m$ dimensional matrices with nonnegative components and $\mathbb{R}_+^n \triangleq \mathbb{R}_+^{n\times 1}$. For a matrix $A \in \mathbb{R}^{m\times n}$, a_{ij} denotes the element located at the ith row and the jth column. Matrix A is said to be positive if $\forall(i,j)\ a_{ij} \geqslant 0$. For any real symmetric matrices P, Q, the notation $P \geq Q$ (respectively, $P > Q$) means that the matrix $P - Q$ is positive semi-definite (respectively, positive definite). The notation $l_2[0,\infty)$ represents the space of square summable infinite vector sequences with the usual norm $\|\cdot\|_2$, that is, a sequence $w = \{w_k\} \in l_2[0,\infty)$ if $\|w\|_2 = \sqrt{\sum_{k=1}^{\infty} w_k^T w_k} < \infty$. $\mathrm{Sys}(A) = A + A^T$ is defined for any matrix $A \in \mathbb{R}^{n\times n}$; The

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superscript "T" denotes matrix transpose and the symbol # is used to represent a matrix which can be inferred by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

Consider the following asymptotically stable system:

$$\Sigma : \begin{cases} x_{k+1} = Ax_k + Bw_k \\ y_k = Cx_k + Dw_k \\ z_k = Lx_k + Gw_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^l$ is the output or measurement, $z_k \in \mathbb{R}^q$ is the signal to be determined. The exogenous disturbance signal $w_k \in \mathbb{R}^m$ is assumed to obey $\{w_k\} \in l_2[0,\infty)$. Furthermore, A, B, C, D, L, and G are real constant matrices with appropriate dimensions. The system (Σ) in (1) is said to be positive if for every $x_0 \in \mathbb{R}_+^n$ and all $w_k \in \mathbb{R}_+^m$, we have $x_k \in \mathbb{R}_+^n$, $y_k \in \mathbb{R}_+^l$, and $z_k \in \mathbb{R}_+^q$ for $k \geq 0$.

To facilitate the subsequent analysis, a characterization of discrete-time positive linear systems is introduced in the following.

Lemma 1 [1]: The system (Σ) in (1) is positive if and only if A, B, C, D, L, and G are positive matrices.

In this note, we aim at obtaining the estimation $\hat{z}_k$ of the signal z_k in (Σ). To be more specific, we are interested in constructing the following filter:

$$\hat{\Sigma} : \begin{cases} \hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}y_k \\ \hat{z}_k = \hat{C}\hat{x}_k + \hat{D}y_k \end{cases} \tag{2}$$

where $\hat{x}_k \in \mathbb{R}^{n_f}$, $0 < n_f \leq n$, and $\hat{z}_k \in \mathbb{R}^q$ for $k \geq 0$. $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are filtering parameters to be determined.

Let $\xi_k = [x_k^T, \hat{x}_k^T]^T$ and $e_k = z_k - \hat{z}_k$. Then, from (1) and (2), the filtering error system can be described by

$$\Sigma_e : \begin{cases} \xi_{k+1} = A_f \xi_k + B_f w_k \\ e_k = C_f \xi_k + D_f w_k \end{cases} \tag{3}$$

where

$$A_f = \begin{bmatrix} A & 0 \\ \hat{B}C & \hat{A} \end{bmatrix}, \quad B_f = \begin{bmatrix} B \\ \hat{B}D \end{bmatrix}$$
$$C_f = [L - \hat{D}C \quad -\hat{C}], \quad D_f = G - \hat{D}D.$$

The purpose of the filter $(\hat{\Sigma})$ for the positive discrete-time system (Σ) is to approximate z_k by $\hat{z}_k$. Therefore, the filter should be chosen in such a way that the estimator $\hat{z}_k$ should be positive, like z_k itself. To ensure the positivity of $\hat{z}_k$, it is natural to require that the filter $(\hat{\Sigma})$ in (2) should be a positive system. Therefore, according to Lemma 1, $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ in (2) should be positive. In addition, the transfer function of system (Σ_e) in (3) is given as

$$\mathcal{G}(z) = C_f(zI - A_f)^{-1}B_f + D_f.$$

Now, we are in a position to formulate the filtering problem to be addressed in this note.

Reduced-Order Positive H_∞ Filtering Problem: Given a disturbance attenuation level $\gamma \in \mathbb{R} > 0$, design a reduced-order positive filter of the form (2) such that the following two requirements are fulfilled.

i) System (Σ_e) in (3) is asymptotically stable with $\|\mathcal{G}\|_\infty < \gamma$, where γ is a prescribed scalar.

ii) Filtering matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are positive.

Remark 1: It can be seen from condition i) that we specify the H_∞ performance of system (3) in terms of its transfer function, that is, we study the positive H_∞ filtering problem under the standard signal space $l_2[0,\infty)$. It will be shown that such a formulation will enable us to obtain necessary and sufficient conditions, which are sufficient to guarantee the same performances under positive inputs. In addition, it follows from condition ii) that the major difference in this note and those in the previous works is that the filter to be designed should be positive, which results in the additional constraints on the filtering matrices. Thus, conventional approaches, using similarity transformation and elimination technique, cannot be applied any more. It is worth pointing out that the filtering error system (Σ_e) in (3) is not a positive system, since it is clear that the state ξ_k still remains positive, whereas the error signals e_k may not stay in the positive orthant. This further indicates that the sign of e_k will not affect that of the estimator $\hat{z}_k$.

We end this section by introducing the following well-known bounded real lemma, which will be used later.

Lemma 2 ([16]): Suppose the filtering error system (Σ_e) in (3) is known, then it is asymptotically stable with $\|\mathcal{G}\|_\infty < \gamma$, if and only if there exists a matrix $P > 0$, such that

$$\begin{bmatrix} A_f^T P A_f - P & A_f^T P B_f & C_f^T \\ \# & B_f^T P B_f - \gamma^2 I & D_f^T \\ \# & \# & -I \end{bmatrix} < 0.$$

III. Main Results

The problem to be discussed in this section is to analyze and synthesize the reduced-order positive H_∞ filter for the positive discrete-time system (Σ). To achieve this, we first formulate the filtering error system (3) into a singular system with an assembled filtering matrix, and then consider the filter analysis and synthesis problem subsequently.

A. System Augmentation Formulation

Observe that matrices A_f, B_f, C_f, and D_f in (3) can be equivalently expressed as

$$A_f = \bar{A} + MK\bar{C}, \quad B_f = \bar{B} + MK\bar{D}$$
$$C_f = \bar{L} + NK\bar{C}, \quad D_f = \bar{G} + NK\bar{D}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 \\ D \end{bmatrix}, \quad K = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$$
$$\bar{L} = [L \quad 0], \quad \bar{G} = G, \quad N = [0 \quad -I].$$

To ensure the positivity of the filter, it can be seen that we only require the matrix K to be positive. On the other hand, even though the filtering matrices have been assembled to the parametric matrix K, one can see that K is still coupled with matrices M and N, which makes it difficult to ensure the positivity of K. In addition, it should be noted that such a problem is generally a bilinear matrix inequality problem, and known to be NP-hard [17]. To tackle the problem, we introduce an auxiliary variable $u_k = K\bar{C}\xi_k + K\bar{D}w_k$, and define $\mathbf{x}_k = [\xi_k^T \ u_k^T]^T$ as a new state variable, then system (Σ_e) in (3) can be described in the form of a singular system as follows:

$$\Sigma_e' : \begin{cases} \mathbf{E}\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}w_k \\ e_k = \mathbf{C}\mathbf{x}_k + \mathbf{D}w_k \end{cases} \tag{4}$$

where

$$\mathbf{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \bar{A} & M \\ K\bar{C} & -I \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \bar{B} \\ K\bar{D} \end{bmatrix}$$
$$\mathbf{C} = [\bar{L} \quad N], \quad \mathbf{D} = \bar{G}.$$

B. Filter Analysis

In this subsection, we shall focus on the reduced-order positive H_∞ filter analysis problem for system (4). We first establish a necessary and sufficient condition to ensure the asymptotic stability of system

(3) with H_∞ performance $\|\mathcal{G}\|_\infty < \gamma$, and then present an equivalent characterization with slack matrices introduced.

Theorem 1: Assume the filtering matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are known. Then, given any matrix $P_2 > 0$, the filtering error system (Σ_e) in (3) is asymptotically stable with $\|\mathcal{G}\|_\infty < \gamma$ if and only if there exists a matrix $P_1 > 0$, a sufficiently large scalar $\alpha > 0$ and a scalar β such that the following matrix inequality holds:

$$\mathcal{A}^T\mathcal{P}\mathcal{A} - \mathcal{Q} < 0 \tag{5}$$

where

$$\mathcal{A} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} \mathbf{E}^T\mathbf{P}\mathbf{E} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \tag{6}$$

with

$$\mathbf{P} \triangleq \begin{bmatrix} P_1 & \beta M P_2 \\ \# & -\alpha P_2 \end{bmatrix}. \tag{7}$$

Proof: (Sufficiency) By applying Schur complement [18] to (5), we have

$$\Omega = \begin{bmatrix} \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathrm{E}^T\mathbf{P}\mathbf{E} & \mathbf{A}^T\mathbf{P}\mathbf{B} & \mathbf{C}^T \\ \# & \mathbf{B}^T\mathbf{P}\mathbf{B} - \gamma^2 I & \mathbf{D}^T \\ \# & \# & -I \end{bmatrix} < 0. \tag{8}$$

Define the transformation matrix as

$$T = \begin{bmatrix} I & 0 & 0 & 0 \\ K\bar{C} & K\bar{D} & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}.$$

Pre- and post-multiplying (8) by T^T and T, respectively, then one can verify that

$$\Sigma \triangleq T^T\Omega T = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \# & \Sigma_{22} \end{bmatrix} < 0 \tag{9}$$

where

$$\Sigma_{11} = \begin{bmatrix} A_f^T P_1 A_f - P_1 & A_f^T P_1 B_f & C_f^T \\ \# & B_f^T P_1 B_f - \gamma^2 I & D_f^T \\ \# & \# & -I \end{bmatrix} \tag{10}$$

$$\Sigma_{12} = \left[(M^T P_1 - \beta P_2 M^T)[A_f \quad B_f], N^T\right]^T \tag{11}$$

$$\Sigma_{22} = -\alpha P_2 + M^T P_1 M - \mathrm{Sys}(\beta M^T M P_2). \tag{12}$$

Obviously, $\Sigma_{11} < 0$ indicates that the filtering error system (Σ_e) in (3) is asymptotically stable with H_∞ performance $\|\mathcal{G}\|_\infty < \gamma$.

(Necessity) If the filtering error system in (3) is asymptotically stable with $\|\mathcal{G}\|_\infty < \gamma$, then according to Lemma 2, we can conclude that there exists a matrix $P_1 > 0$, such that $\Sigma_{11} < 0$, where Σ_{11} is defined in (10).

Therefore, given any matrix $P_2 > 0$, and a real scalar β, for a sufficiently large scalar $\alpha > 0$, we have

$$\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12} < 0, \tag{13}$$

which together with Schur complement equivalence, further indicates that (5) holds. This completes the proof. □

Remark 2: A major disadvantage of existing LMI characterizations is that the Lyapunov matrix used to check the H_∞ performance is closely related to the filtering matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$, which makes them difficult to solve. This is particularly critical when the positivity constraint on these filtering matrices is taken into account. However, based on system augmentation formation before, these matrices are firstly assembled to a new matrix K, which is no longer coupled with the Lyapunov matrix P_1, and can be further parametrized by a completely free matrix $P_2 > 0$.

In the following, we present an equivalent characterization with slack matrices introduced for the H_∞ filter analysis on the basis of Theorem 1.

Theorem 2: Assume the filtering matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are known. Then, given any matrix $P_2 > 0$, the filtering error system (Σ_e) in (3) is asymptotically stable with $\|\mathcal{G}\|_\infty < \gamma$ if and only if there exist matrices $P_1 > 0$, F_i, $H_i (i = 1, 2, \ldots, 6)$, a sufficiently large scalar $\alpha > 0$ and a scalar β such that the following matrix inequality holds:

$$\begin{bmatrix} -\mathcal{Q} + \mathcal{A}^T\mathcal{F} + \mathcal{F}^T\mathcal{A} & -\mathcal{F}^T + \mathcal{A}^T\mathcal{H} \\ \# & \mathcal{P} - \mathcal{H} - \mathcal{H}^T \end{bmatrix} < 0 \tag{14}$$

where $\mathcal{A}$, $\mathcal{P}$ and $\mathcal{Q}$ are defined in (6), and

$$\mathcal{F} = \begin{bmatrix} F_1 & F_2 & F_3 \\ -\alpha P_2 K\bar{C} & \alpha P_2 & -\alpha P_2 K\bar{D} \\ F_4 & F_5 & F_6 \end{bmatrix}$$

$$\mathcal{H} = \begin{bmatrix} H_1 & H_2 & H_3 \\ \beta P_2 M^T & 0 & 0 \\ H_4 & H_5 & H_6 \end{bmatrix}.$$

Proof: (Sufficiency) Pre- and post-multiplying (14) by $[I \ \mathcal{A}^T]$ and its transpose yields (5). Thus, according to Theorem 1, we can conclude that the filtering error system (Σ_e) in (3) is asymptotically stable with H_∞ performance $\|\mathcal{G}\|_\infty < \gamma$.

(Necessity) Select

$$\mathcal{F} = \begin{bmatrix} \beta M P_2 K\bar{C} & -\beta M P_2 & \beta M P_2 K\bar{D} \\ -\alpha P_2 K\bar{C} & \alpha P_2 & -\alpha P_2 K\bar{D} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{H} = \begin{bmatrix} P_1 & 0 & 0 \\ \beta P_2 M^T & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

then it can be easily checked that $\mathcal{P} - \mathcal{H} - \mathcal{H}^T < 0$, and

$$\begin{bmatrix} -\mathcal{Q} + \mathcal{A}^T\mathcal{F} + \mathcal{F}^T\mathcal{A} & -\mathcal{F}^T + \mathcal{A}^T\mathcal{H} \\ \# & \mathcal{P} - \mathcal{H} - \mathcal{H}^T \end{bmatrix} = \begin{bmatrix} I & -\mathcal{A}^T \\ 0 & I \end{bmatrix} \times \begin{bmatrix} -\mathcal{Q} + \mathcal{A}^T\mathcal{P}\mathcal{A} & 0 \\ 0 & \mathcal{P} - \mathcal{H} - \mathcal{H}^T \end{bmatrix} \times \begin{bmatrix} I & 0 \\ -\mathcal{A} & I \end{bmatrix}.$$

Therefore, if the filtering error system (Σ_e) in (3) is asymptotically stable with $\|\mathcal{G}\|_\infty < \gamma$, according to Theorem 1, we have $-\mathcal{Q} + \mathcal{A}^T\mathcal{P}\mathcal{A} < 0$, which further implies that there exist matrices $P_1 > 0$, $P_2 > 0$, F_i, $H_i (i = 1, 2, \ldots, 6)$, a sufficiently large scalar $\alpha > 0$ and a scalar β such that (14) holds. This completes the proof. □

C. Filter Synthesis

In this subsection, based on Theorem 2, we are ready to deal with the design problem for the reduced-order positive H_∞ filter, a necessary and sufficient condition is obtained for the existence of a positive filter, and an iterative LMI algorithm is developed to obtain a desired filter.

Theorem 3: For the positive discrete-time system (Σ) in (1) and a given scalar $\gamma > 0$, the reduced-order positive H_∞ filtering problem is solvable by a positive filter $(\hat{\Sigma})$ in (2) if and only if there exists a matrix $P_1 > 0$, a diagonal matrix $P_2 > 0$, matrices R, U, V, F_i,

$H_i (i = 1, 2, \ldots, 6)$, a sufficiently large scalar $\alpha > 0$ and a scalar β such that (15) and (16) hold,

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \text{ with } R_{11} \in \mathbb{R}_+^{n_f \times n_f}$$
$$R_{12} \in \mathbb{R}_+^{n_f \times l}, \ R_{21} \in \mathbb{R}_+^{q \times n_f}, \ R_{22} \in \mathbb{R}_+^{q \times l} \quad (15)$$

with (16) as shown at the bottom of the page, where

$$\begin{aligned}
\Pi_{11} &= -P_1 + 2\alpha U^T P_2 U \\
&\quad + \mathrm{Sys}(\bar{A}^T F_1 + \bar{L}^T F_4 - 2\alpha \bar{C}^T R^T U) \\
\Pi_{12} &= \bar{A}^T F_2 + \bar{L}^T F_5 + F_1^T M + F_4^T N + 2\alpha \bar{C}^T R^T \\
\Pi_{13} &= \bar{A}^T F_3 + \bar{L}^T F_6 + F_1^T \bar{B} + F_4^T \bar{G} \\
&\quad - 2\alpha \bar{C}^T R^T V - 2\alpha U^T R \bar{D} + 2\alpha U^T P_2 V \\
\Pi_{14} &= -F_1^T + \bar{A}^T H_1 + \beta \bar{C}^T R^T M^T + \bar{L}^T H_4 \\
\Pi_{22} &= \mathrm{Sys}(M^T F_2 + N^T F_5 - \alpha P_2) \\
\Pi_{23} &= M^T F_3 + N^T F_6 + F_2^T \bar{B} + F_5^T \bar{G} + 2\alpha R \bar{D} \\
\Pi_{24} &= -F_2^T + M^T H_1 - \beta P_2 M^T + N^T H_4 \\
\Pi_{33} &= -\gamma^2 I + 2\alpha V^T P_2 V \\
&\quad + \mathrm{Sys}(\bar{B}^T F_3 + \bar{G}^T F_6 - 2\alpha \bar{D}^T R^T V) \\
\Pi_{34} &= -F_3^T + \bar{B}^T H_1 + \bar{G}^T H_4 + \beta \bar{D}^T R^T M^T \\
\Pi_{44} &= P_1 - \mathrm{Sys}(H_1).
\end{aligned}$$

In this case, the parameters of the desired filter $(\hat{\Sigma})$ are obtained as

$$K = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = P_2^{-1} R. \quad (17)$$

Proof: (Sufficiency) Note that $P_2 > 0$ is diagonal, if follows from (15) and (17) that K is positive, which indicates that the filtering system $(\hat{\Sigma})$ is positive. By expanding the left side of (14), and noting that, for any U and V

$$\begin{aligned}
-2\alpha \varsigma^T K^T P_2 K \varsigma &\leq -2\alpha \varsigma^T K^T P_2 K \varsigma \\
&\quad + 2\alpha (\upsilon - K\varsigma)^T P_2 (\upsilon - K\varsigma) \\
&= -2\alpha \varsigma^T K^T P_2 \upsilon - 2\alpha \upsilon^T P_2 K \varsigma \\
&\quad + 2\alpha \upsilon^T P_2 \upsilon
\end{aligned} \quad (18)$$

where

$$\varsigma = [\bar{C} \quad 0 \quad \bar{D}], \quad \upsilon = [U \quad 0 \quad V]$$

we have (14) holds if (16) holds, where the parametrization $R = P_2 K$ is applied.

(Necessity) If the reduced-order positive H_∞ filtering problem is solvable by a positive filter $(\hat{\Sigma})$ in (2), then according to Theorem 2, there exists a matrix $P_1 > 0$, a diagonal matrix $P_2 > 0$, matrices $F_i (i = 1, 2, \ldots, 6)$, $H_i (i = 1, 2, \ldots, 4)$, a sufficiently large scalar $\alpha > 0$ and a scalar β such that (14) holds. Note that equality holds in (18) when $U = K\bar{C}$, $V = K\bar{D}$. Therefore, if we choose $U = K\bar{C}$, $V = K\bar{D}$, then it can be checked from (14), as well as the fact $R = P_2 K$, that (15) and (16) hold. This completes the proof. □

Remark 3: It should be emphasized that the approach developed in our note is particularly flexible to parametrize the matrix K. From the proof in Theorem 3, one can see that matrix K to be designed is coupled with the slack matrix P_2, while it is independent of the Lyapunov matrix P_1. Such an approach not only reduces the conservatism in general, but also introduces the flexibility to the design of filtering matrices when additional design specifications are involved. To be specific, one may use P_2 to parametrize K when the positivity of matrix K is imposed, and matrix P_2 can be chosen as any positive diagonal matrix. In general, one can see that the positivity of matrix K can be imposed when it is parameterized by the positive definite matrix P_2 provided that P_2^{-1} is positive in Theorem 3.

Although the inequality in (16) is not a linear matrix inequality with respect to the parameters $P_1, P_2, R, U, V, F_i, H_i (i = 1, 2, \ldots, 6), \alpha, \beta$, it can be easily observed that if α, β, U, and V are fixed in (16), then it reduces to an LMI problem with respect to the remaining parameters, which is convex and can be solved efficiently by existing software [18].

Now, we are in a position to illustrate how α, β, U, and V can be selected properly to design the filter. From the proof of necessity in Theorem 1, it can be easily seen that the larger the α, the less conservative is the condition. Thus, if there does not exist a sufficiently large $\alpha > 0$ such that (16) holds, then it is reasonable to conclude that a solution to the reduced-order positive H_∞ filtering problem may not exist. Also, the scalar β can be chosen arbitrarily with the precondition that the scalar α is sufficiently large, which will introduce the flexibility when considering the synthesis problem of the reduced-order positive H_∞ filter. The remaining problem is then how to choose U and V. It can be seen that, from the proof of Theorem 3, the scalar ε satisfying $\Pi(\alpha, \beta, U, V) < \varepsilon I$ achieves its minimum when $U = P_2^{-1} R \bar{C}$, and $V = P_2^{-1} R \bar{D}$, which points to an iterative approach to solve (16).

Based on the above discussions, we propose the following algorithm.

Iterative LMI Algorithm:

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1) Set $j = 1$, α being a sufficiently large value and β is some scalar. Find two initial matrices U_1 and V_1 such that the following system:

$$\begin{cases} \xi_{k+1} = (\bar{A} + M U_1)\xi_k + (\bar{B} + M V_1) w_k \\ e_k = (\bar{L} + N U_1)\xi_k + (\bar{G} + N V_1) w_k, \end{cases} \quad (19)$$

achieves the H_∞ performance with $\|e\|_{l_2} < \gamma \|w\|_{l_2}$.

2) For fixed α, β, U_j, and V_j, solve the following convex optimization problem for the parameters in $\Omega_j \triangleq \{P_{1j} > 0, P_{2j} > 0 \text{ is diagonal}, R_j, F_{ij}, H_{ij} (i = 1, 2, \ldots, 6)\}$:

$$\min_{\Omega_j} \varepsilon_j$$
$$\text{subject to} \begin{cases} R_j \in \mathbb{R}^{(n_f + l) \times (n_f + q)} \text{ is positive} \\ \Pi(\alpha, \beta, U_j, V_j) < \varepsilon_j I. \end{cases}$$

Denote ε_j^* as the minimized value of ε_j.

$$\Pi(\alpha,\beta,U,V) = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \alpha \bar{C}^T R^T + \bar{A}^T H_2 + \bar{L}^T H_5 & -F_4^T + \bar{A}^T H_3 + \bar{L}^T H_6 \\ \# & \Pi_{22} & \Pi_{23} & \Pi_{24} & -\alpha P_2 + M^T H_2 + N^T H_5 & -F_5^T + M^T H_3 + N^T H_6 \\ \# & \# & \Pi_{33} & \Pi_{34} & \alpha \bar{D}^T R^T + \bar{B}^T H_2 + \bar{G}^T H_5 & -F_6^T + \bar{B}^T H_3 + \bar{G}^T H_6 \\ \# & \# & \# & \Pi_{44} & -H_2 & -H_3 - H_4^T \\ \# & \# & \# & \# & -\alpha P_2 & -H_5 \\ \# & \# & \# & \# & \# & I - \mathrm{Sys}(H_6) \end{bmatrix} < 0 \quad (16)$$

3) If $\varepsilon_j^* \leq 0$, then a desired parametric matrix K is obtained as (17). STOP. If not, then go to Step 4.
4) If $|\varepsilon_j^* - \varepsilon_{j-1}^*| \leq \delta$, where δ is a prescribed tolerance, then go to Step 5. If not, update U_{j+1} and V_{j+1} as

$$U_{j+1} = P_{2j}^{-1} R_j \bar{C}, \quad V_{j+1} = P_{2j}^{-1} R_j \bar{D}.$$

Set $j = j + 1$, then go to Step 2.
5) A solution to the reduced-order positive H_∞ filtering problem may not exist. STOP (or select a larger α, some β, and other initial values U_1 and V_1 to run the algorithm again).

Before ending this section, we present several remarks on the algorithm as follows:

Remark 4: Notice that it is the relative magnitude between α and β which is important in the running of the algorithm. One may choose typical values of β as $\beta = 0$ (or -1, 1) while keeping α sufficiently large. For the choice of α, according to our numerical experience, α should be selected among the range $10^3 \sim 10^6$.

Remark 5: For the computation of the initial matrices U_1 and V_1 in Step 1, one can utilize existing approaches to find them, and details on this issue are omitted here due to its simplicity [13], [19]. If no such matrices exist, then we can conclude immediately that there does not exist a solution to the reduced-order positive H_∞ filtering problem proposed in Section II.

In addition, it can be seen that if, for some K^*, $P_1^* > 0$, diagonal $P_2^* > 0$, F_i^*, $H_i^*(i = 1, 2, \ldots, 6)$, $\alpha^* > 0$, and β^*, (18) holds, which means that there exists a solution to the positive H_∞ filtering problem, then (16) will also be feasible, i.e., $\Pi(\alpha^*, \beta^*, U, V) < 0$, provided that $2\alpha^*(\upsilon - K^*\varsigma)^T P_2^*(\upsilon - K^*\varsigma)$ is "small" enough. Thus, the solvability of the algorithm can be further improved by choosing U_1 and V_1 such that $\|2\alpha^*(\upsilon_1 - K^*\varsigma)^T P_2^*(\upsilon_1 - K^*\varsigma)\|$ is small enough, where $\upsilon_1 = [U_1 \ 0 \ V_1]$. Since the reduction of α^* contradicts the requirement of α^* being sufficiently large, an alternative way is to make $\|\upsilon_1 - K^*\varsigma\|$ sufficiently small.

Remark 6: Note the fact that the sequence ε_j^* is monotonically non-increasing with j in Step 2, that is, $\varepsilon_{j+1}^* \leq \varepsilon_j^*$. Therefore, even though the convergence of the algorithm is not guaranteed, it is possible that, after a sufficiently large number of iterations, ε_j^* may converge to a non-positive number, which corresponds to Step 3. As a result, we can conclude that there exists a solution to the reduced-order positive H_∞ filtering problem, although ε_j^* may not always converge to a minimum.

IV. Numerical Simulation

In this section, we provide a simulation example to illustrate the application of the proposed method for filter design in this note.

Consider the following Leslie model describing a certain pest's structured population dynamics, which was invented by and named after P. H. Leslie:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} f_1 & f_2 & f_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix} x_k + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} w_k \\ y_k &= \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} x_k + \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} w_k \end{aligned} \tag{20}$$

where $x_k = [x_k^1 \ x_k^2 \ x_k^3]^T$ with x_k^1, x_k^2 and x_k^3 representing the number of juvenile pests, immature pests and adult pests at time k, respectively. Here, f_i represents the birth rate for parents in age class i, and s_i denotes the rate of individuals that survive from age class i to age class $i + 1$. The structure of the input matrix indicates that all pests are affected by external disturbance w_k. The structure of the output matrix indicates that the number of immature and adult pests can be measured with external disturbance added.

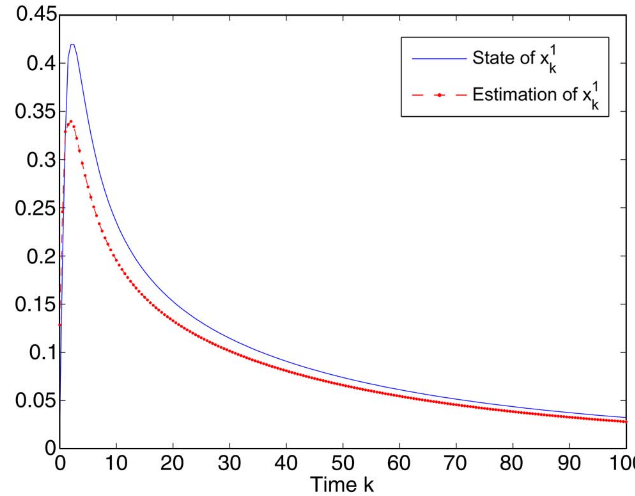

Fig. 1. State x_k^1 and its estimation.

Assume that the parameters for this model are given as

$$\begin{aligned} f_1 &= 0.1595, & f_2 &= 0.1890, & f_3 &= 0.2713 \\ s_1 &= 0.5091, & s_2 &= 0.6740, & b_1 &= 0.1350 \\ b_2 &= 0.0128, & b_3 &= 0.0128, & b_4 &= 0.0510 \\ b_5 &= 0.1021, & b_6 &= 0.1250, & c &= 1.0000 \\ d_1 &= d_4 = 0, & d_2 &= 0.1250, & d_3 &= 0.1460. \end{aligned}$$

To estimate the quantities of juvenile pests, i.e., $z_k = x_k^1 = [1 \ 0 \ 0]x_k$, one can design a positive first-order filter as follows:

$$\begin{cases} \hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}y_k \\ \hat{z}_k = \hat{C}\hat{x}_k + \hat{D}y_k. \end{cases} \tag{21}$$

We assume that the H_∞ performance bound is $\gamma = 0.15$, by implementing *Iterative LMI Algorithm* via Yalmip [20] with $\alpha = 10^5$ and $\beta = 0$, one can find that the condition in Theorem 3 is feasible with the following solution:

$$\begin{aligned} P_2 &= \begin{bmatrix} 1.01260 & 0 \\ 0 & 1.45325 \end{bmatrix} \\ R &= \begin{bmatrix} 0.23106 & 0.00003 & 0.00003 \\ 0.20534 & 0.25998 & 0.49998 \end{bmatrix}. \end{aligned}$$

Thus, the positive filtering matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ in (21) can be readily obtained as

$$\begin{aligned} \hat{A} &= 0.22819, & \hat{B} &= [0.00003 \quad 0.00003] \\ \hat{C} &= 0.14130, & \hat{D} &= [0.17889 \quad 0.34404]. \end{aligned}$$

Also, it can be verified easily that the actual H_∞ performance of the filtering error system is 0.1415, which compares well with the value $\gamma = 0.15$ adopted. Fig. 1 is the simulation result of the actual state of x_k^1 and its estimation, where the initial condition of the error system is $[0.03, 0.08, 0.10, 0.05]^T$, and the exogenous disturbance input is given as $w_k = [1/(1 + 0.25k), \ e^{-0.02k}]^T$.

V. Conclusion

In this note, the problem of positive reduced-order filtering with H_∞ performance for positive linear discrete-time systems has been studied based on an augmentation approach. New characterizations on the H_∞ performance of the filtering error system have been established in terms of matrix inequalities. A necessary and sufficient condition is presented to verify the existence of the desired positive filter, and an iterative linear matrix inequality algorithm is proposed to compute the filtering matrices, which can be parameterized by a positive definite matrix independent of the Lyapunov matrix. The effectiveness of the derived condition has been demonstrated by an illustrative example.

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A Negative Imaginary Lemma and the Stability of Interconnections of Linear Negative Imaginary Systems

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***Abstract*—The note is concerned with linear negative imaginary systems. First, a previously established Negative Imaginary Lemma is shown to remain true even if the system transfer function matrix has poles on the imaginary axis. This result is achieved by suitably extending the definition of negative imaginary transfer function matrices. Secondly, a necessary and sufficient condition is established for the internal stability of the positive feedback interconnections of negative imaginary systems. Meanwhile, some properties of linear negative imaginary systems are developed. Finally, an undamped flexible structure example is presented to illustrate the theory.**

***Index Terms*—Linear systems, negative imaginary systems, positive real systems, stability.**

I. INTRODUCTION

Systems which dissipate energy often lead to positive real systems. The study of positive real systems has achieved great successes both in theory and in practice [1], [2]. The positive real property may be seen as a generalization of the positive definite property of matrices to the case of dynamic systems [2], where only the real part of the transfer function matrix is considered. Positive real systems have many uses in practice. For instance, they can be realized with electrical circuits using only resistors, inductors and capacitors [1]. For mechanical positive real systems, velocity sensors and force actuators can be used to implement a control system with a guarantee of closed-loop stability.

One major limitation of positive real systems is that their relative degree must be zero or one [2]. This limits the application of positive real theory. For example, a lightly damped flexible structure with a collocated velocity sensor and force actuator can typically be modeled by a sum of second-order transfer functions as $G(s) = \sum_{i=0}^{\infty}(\psi_i^2 s/(s^2 + 2\zeta_i\omega_i s + \omega_i^2))$, where ω_i is the mode frequency, $\zeta_i > 0$ is the damping coefficient associated of the i-th mode, and ψ_i is determined by the boundary condition on the underlying partial differential equation. In some cases (for example, when using piezoelectric sensors), the sensor output is proportional to position rather than velocity. In this case, the transfer function $G(s)$ given above is the transfer function from the actuator input to the derivative of the sensor output. In the case of a lightly damped flexible structure with a collocated position sensor and force actuator, the transfer function will be of the form $G(s) = \sum_{i=0}^{\infty}(\psi_i^2/(s^2 + 2\zeta_i\omega_i s + \omega_i^2))$. It can be seen that in this case, the relative degree of the system is more than unity. Hence, the standard positive real theory will not be helpful in establishing closed-loop stability. However, such a transfer function would satisfy the following negative imaginary condition $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$. Such systems are called "systems with negative imaginary frequency

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