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TECHNICAL NOTE

Relating the maximum radial stress on pile shaft to pile base resistance

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An approximate analytic relationship is developed between the maximum radial stress on the shaft of a displacement pile in sand and the base resistance of the pile. Using the cavity expansion analogy, together with a confined failure mechanism, the ratio between the two quantities, defined as a factor S_t, is established as a function of the friction angle, shear stiffness, compressibility and mean effective stress of the sand near the pile tip. It is shown that, given otherwise identical input parameters, the value of S_t will decrease with increasing friction angle, and with decreasing mean stress level. It also tends to decrease with an increase in relative density. It is predicted that S_t has typical values between 0·03 and 0·05, in broad agreement with the range of empirically derived values in the literature. The relationship also predicts that S_t may take much higher values (~0·1) for piles installed in dense sand or in highly compressible sand. Because of the analytical nature, the established relationship provides useful insights into the mechanisms involved and important implications for design practice.

KEYWORDS: bearing capacity; compressibility; piles; sands; stiffness

La présente communication développe des rapports d'analyse approximatifs entre les contraintes radiales maximales appliquées sur le fût d'un pieu à déplacement dans le sable et la résistance de base de ce pieu. En utilisant l'analogie de l'expansion de cavité, ainsi qu'un mécanisme de rupture renfermé, on établit le rapport entre les deux quantités, défini comme un facteur S_t, en fonction de l'angle de frottement, de la rigidité au cisaillement, de la compressibilité et de la tension efficace moyenne du sable à proximité de la pointe du pieu. On démontre qu'en présence de paramètres d'entrée à tous autres égards identiques, la valeur de S_t diminue au fur et à mesure de l'augmentation de l'angle de frottement et de la diminution de la contrainte moyenne. En outre, elle a également tendance à diminuer sous l'effet d'une augmentation de la masse volumique. On est en mesure de prédire que les valeurs de S_t sont généralement comprises entre 0,03 et 0,05, et correspondent dans l'ensemble à la plage de valeurs dérivées de façon empirique, dans des ouvrages pertinents. Ce rapport permet également de prédire que les valeurs de S_t peuvent être beaucoup plus élevées (~0·1) lorsque les pieux sont installés dans du sable dense ou du sable extrêmement compressible. Compte tenu de la nature analytique, le rapport établi fournit un aperçu utile sur les mécanismes en présence et des implications importantes pour la pratique du design.

INTRODUCTION

The shaft resistance of displacement piles in sand has been an area of great uncertainty, and thus of considerable interest, in foundation design. Recent experiments with instrumented model piles in the field (Lehane *et al.*, 1993; Chow, 1997), through measurement of radial effective stresses acting on the pile shaft, have significantly improved understanding of shaft friction characteristics. This has allowed the development of new design approaches with increased rationality (Randolph *et al.*, 1994; Jardine *et al.*, 2005). These new approaches, while presented in different forms, share two important considerations (Fig. 1): (a) a maximum shaft friction, associated with a maximum radial effective stress, exists in the vicinity of the pile tip; and (b) a degradation of the maximum shaft friction or the maximum radial effective stress will occur as the pile tip advances further. The physical basis for friction degradation has been revealed by the aforementioned model tests in the field, and later by model tests on the centrifuge (Klotz & Coop, 2001; White & Lehane, 2004). With respect to the maximum shaft friction, however, the factors governing its characteristics remain unclear. This may be due to the complexity of pile–soil interactions in the highly stressed zone near the tip, and to the lack of reliable data in this zone.

In light of the work of Vesic (1970) and Fleming *et al.* (1992), Randolph *et al.* (1994) made a good proposal relating the maximum shaft friction, τ_{max}, to the pile base resistance, q_b

$$\frac{\tau_{max}}{q_b} = S_t \tan\delta \tag{1}$$

where δ is the interface friction angle between the pile and the soil, and S_t is the ratio between the maximum radial effective stress $\sigma'_{r,max}$ and the base resistance (Fig. 1). This can be shown by assuming that the Coulomb failure criterion applies

$$\sigma'_{r,max} = S_t q_b \tag{2}$$

Fleming *et al.* (1992) suggested a constant value 0·02 for S_t. Later, Randolph *et al.* (1994) proposed an exponential expression relating S_t to the friction angle of the sand near the tip, ϕ, as

$$S_t = a \exp(-b \tan\phi) \tag{3}$$

where a and b are two parameters requiring back-analysis of pile test results. They suggested that $a = 2$ and $b = 7$, and predicted that S_t values are between 0·02 and 0·05 for a range of friction angles (27–33°).

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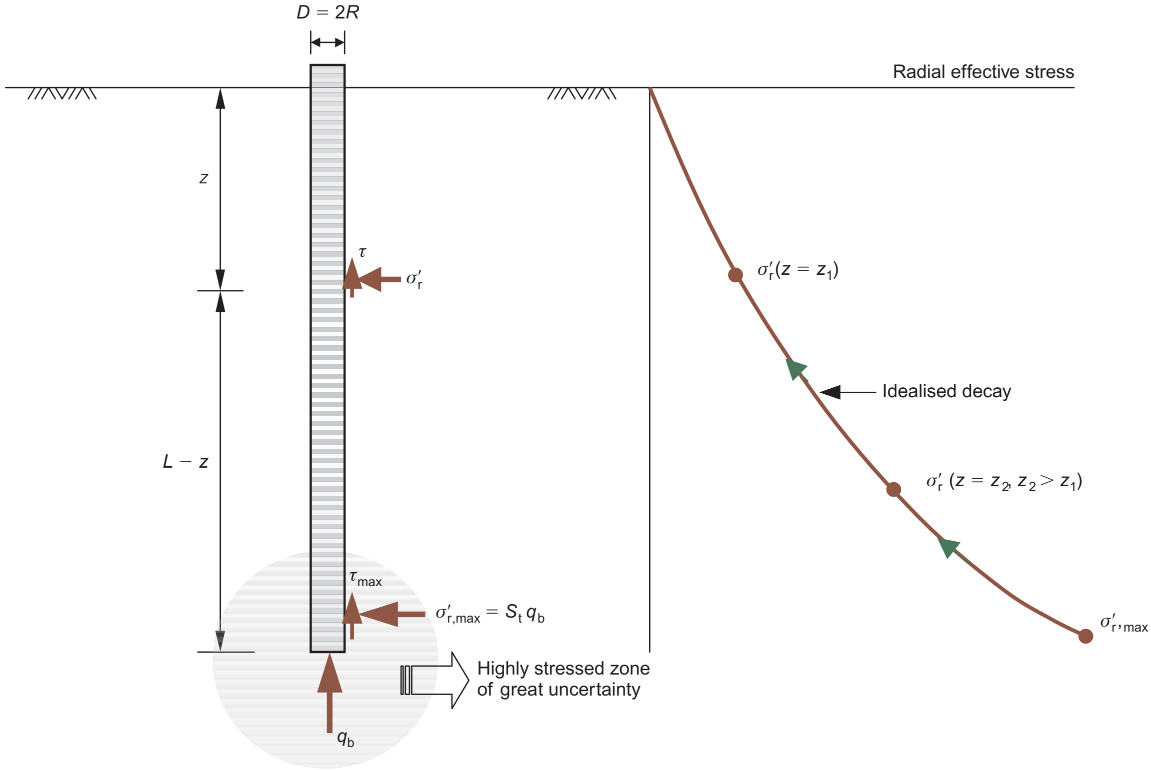

Fig. 1. Schematic illustration of varying radial effective stresses along pile shaft

Because of the empirical nature, the two parameters in equation (3) do not bear physical meanings, and their values depend on the database used as well as on the interpretations. Keeping in mind the physical process involved in pile installation, a reasonable postulation made here is that S_t should be closely linked with the soil properties near the pile tip, such as shearing resistance, stiffness, relative density and confining stress level. In this respect, an expression for S_t that is able to account for these key factors in a rational manner is much preferred, since it may provide insights into the problem, and lead to improved understanding. This is the motivation of the present study.

MODELLING

The spherical cavity expansion analogy has been well accepted for analysis of piles and penetrometers (e.g. Vesic, 1972; Yu & Houlsby, 1991; Yasufuku & Hyde, 1995). A comparison of various failure patterns (Yang *et al.*, 2005) suggests that the confined local failure mechanism, shown in Fig. 2, can provide a fairly reasonable prediction of the end-bearing capacity of displacement piles in sand. It is assumed that the limit pressure acts on the spherical surface AC, and that ACF forms part of the wedge under the pile, with the angle ψ equal to $(\pi/4 + \phi/2)$.

At the limit state the cavity has a radius R_u and the plastic zone extends to a radius R_p, beyond which the soil mass remains in a state of elastic equilibrium. By combining the equilibrium condition, equation (4), with the Coulomb yield criterion, equation (5), the following is obtained

$$\frac{\partial \sigma'_r}{\partial r} + 2\frac{\sigma'_r - \sigma'_\theta}{r} = 0 \tag{4}$$

$$\sigma'_r(1 - \sin\phi) = (1 + \sin\phi)\sigma'_\theta \tag{5}$$

where σ'_r and σ'_θ are the radial and circumferential stress components. The radial stress in the plastic zone, σ'_{rp}, can be derived as

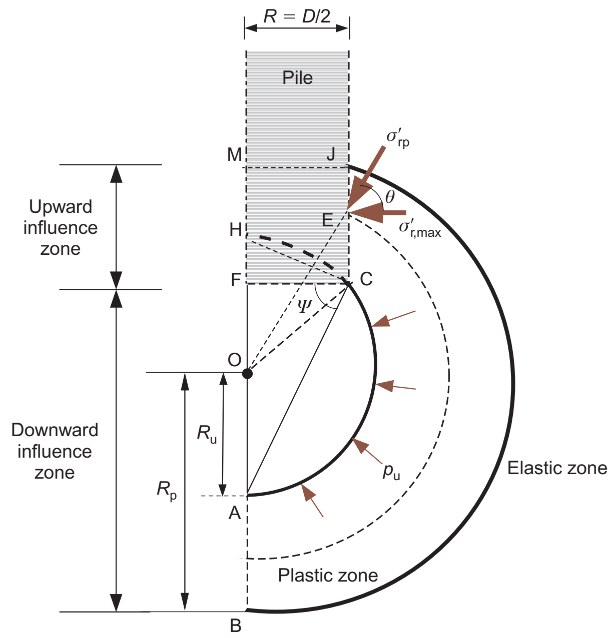

Fig. 2. Cavity expansion model

$$\sigma'_{rp} = p_u \left(\frac{R_u}{r}\right)^{4\sin\phi/(1+\sin\phi)} \tag{6}$$

where r is a radial distance varying from R_u to R_p.

Instrumented pile tests have indicated that the maximum radial stress occurs in a zone close to the pile tip. Jardine *et al.* (2005) assume that the maximum radial stress is at the position $4D$ (D is pile diameter) above the pile tip, whereas Lehane *et al.* (2005) assume that it is at $2D$ above the pile tip. Given this uncertainty, and considering that the sand close to the pile tip is at failure, a reasonable assumption made here is that the maximum radial stress acts at some distance from the pile tip in the plastic zone (point E shown in Fig. 2). The distance (CE) is taken as λD, where λ is a proportional factor to be discussed later. Without involving the complicated conversion from spherical to cylindrical stress components, the maximum radial stress on the pile is approximately estimated here as

$$\sigma'_{\mathrm{r,max}} = \sigma'_{\mathrm{rp}} \cos\theta \tag{7}$$

For most practical cases of interest the above expression provides a reasonable level of accuracy: the difference is within about 5% compared with the more complicated expression. Making reference to the triangle OCE, and denoting the angle formed by CO and CE as β, one has

$$\cos\theta = \sin\beta \frac{R_{\mathrm{u}}}{\mathrm{OE}} \tag{8}$$

where OE, representing the radial distance r, can be determined as

$$\mathrm{OE} = r = \sqrt{(\lambda D)^2 + R_{\mathrm{u}}^2 - 2\lambda D R_{\mathrm{u}} \cos\beta} \tag{9}$$

Based on equations (6)–(9), and noting that $\beta = (\pi/2 + \phi)$, the maximum radial stress can be established in the form

$$\sigma'_{\mathrm{r,max}} = p_{\mathrm{u}} \cos\phi \times \left\{ \frac{R_{\mathrm{u}}}{\sqrt{(\lambda D)^2 + R_{\mathrm{u}}^2 - 2\lambda D R_{\mathrm{u}} \cos[\phi + (\pi/2)]}} \right\}^{4\sin\phi/(1+\sin\phi)+1} \tag{10}$$

Now, introducing the relationship between cavity pressure p_{u} and pile base resistance q_{b} as (Yasufuku & Hyde, 1995)

$$q_{\mathrm{b}} = \frac{p_{\mathrm{u}}}{1 - \sin\phi} \tag{11}$$

and noting that the radius of the cavity is given as (Yang, 2006)

$$R_{\mathrm{u}} = \frac{D}{2\cos\phi} \tag{12}$$

equation (10) can be further written as

$$\sigma'_{\mathrm{r,max}} = q_{\mathrm{b}}(1 - \sin\phi)\cos\phi \times \left(4\lambda^2 \cos^2\phi + 2\lambda \sin 2\phi + 1\right)^{-2\sin\phi/(1+\sin\phi)-1/2} \tag{13}$$

It then follows that

$$S_{\mathrm{t}} = \frac{\sigma'_{\mathrm{r,max}}}{q_{\mathrm{b}}} = (1 - \sin\phi)\cos\phi \left(4\lambda^2 \cos^2\phi + 2\lambda \sin 2\phi + 1\right)^{-2\sin\phi/(1+\sin\phi)-1/2} \tag{14}$$

With equation (14), a preliminary evaluation of S_{t} can be made, as shown in Table 1 for various combinations of ϕ and λ. Given a range of friction angles (29–35°), the value of S_{t} varies from 0·06 to 0·04 for $\lambda = 1$ and from 0·02 to 0·01 for $\lambda = 2$. It is striking that the predicted values are in broad agreement with the range of empirically derived values in the literature.

Having noted the influence of the parameter λ and the existing uncertainty with the location of the maximum radial stress, a rational consideration taken here is to average S_{t} over the plastic zone (i.e. from point C to point J in Fig. 2) such that

$$\overline{S}_{\mathrm{t}} = \frac{1}{I_{\mathrm{FU}}} \int_0^{I_{\mathrm{FU}}} S_{\mathrm{t}} \mathrm{d}(\lambda D) \tag{15}$$

where $I_{\mathrm{FU}} = \mathrm{FM}$ denotes the upper limit of the influence zone (Yang, 2006)

$$I_{\mathrm{FU}} = \frac{D}{2}\left(\sqrt{\frac{\xi^2}{\cos^2\phi} - 1} - \tan\phi\right) \tag{16}$$

in which

$$\xi = \frac{R_{\mathrm{p}}}{R_{\mathrm{u}}} = \sqrt[3]{\frac{I_{\mathrm{r}}}{1 + I_{\mathrm{r}}\Delta}} \tag{17a}$$

and

$$I_{\mathrm{r}} = \frac{G}{p'_0 \tan\phi} \tag{17b}$$

Here I_{r} is known as the rigidity index, Δ is the average volumetric strain in the plastic zone, G is the shear modulus, and p'_0 is the mean effective stress at the pile tip. With equations (14)–(16), the factor S_{t} is finally given as (with the upper bar removed for convenience)

$$S_{\mathrm{t}} = \frac{1}{\chi}\int_0^{\chi} (1 - \sin\phi)\cos\phi \times \left(4\lambda^2 \cos^2\phi + 2\lambda \sin 2\phi + 1\right)^{-2\sin\phi/(1+\sin\phi)-1/2} \mathrm{d}\lambda \tag{18}$$

where $\chi = I_{\mathrm{FU}}/D$ is a dimensionless parameter.

The expression established above makes it possible to investigate how S_{t} varies with key soil properties. Such an investigation is of considerable interest, in that it can help identify the governing factors for the relationship between the maximum radial stress and the pile base resistance, and thereby provide useful design implications, as will be discussed in the next section.

PREDICTION AND DISCUSSION

Figure 3 presents calculated S_{t} values as a function of the friction angle for piles in medium dense sand ($D_{\mathrm{r}} = 50\%$)

Table 1. Preliminary estimates of S_{t}

	ϕ										
	25°	27°	29°	31°	33°	35°	37°	39°	41°	43°	45°
$\lambda = 1$	0·0762	0·0675	0·0599	0·0533	0·0475	0·0424	0·0379	0·0339	0·0304	0·0272	0·0244
$\lambda = 2$	0·0233	0·0202	0·0176	0·0154	0·0136	0·0120	0·0107	0·0095	0·0085	0·0076	0·0069
$\lambda = 3$	0·0106	0·0091	0·0078	0·0068	0·0059	0·0052	0·0046	0·0040	0·0036	0·0032	0·0029

Note: All values are calculated using equation (14).

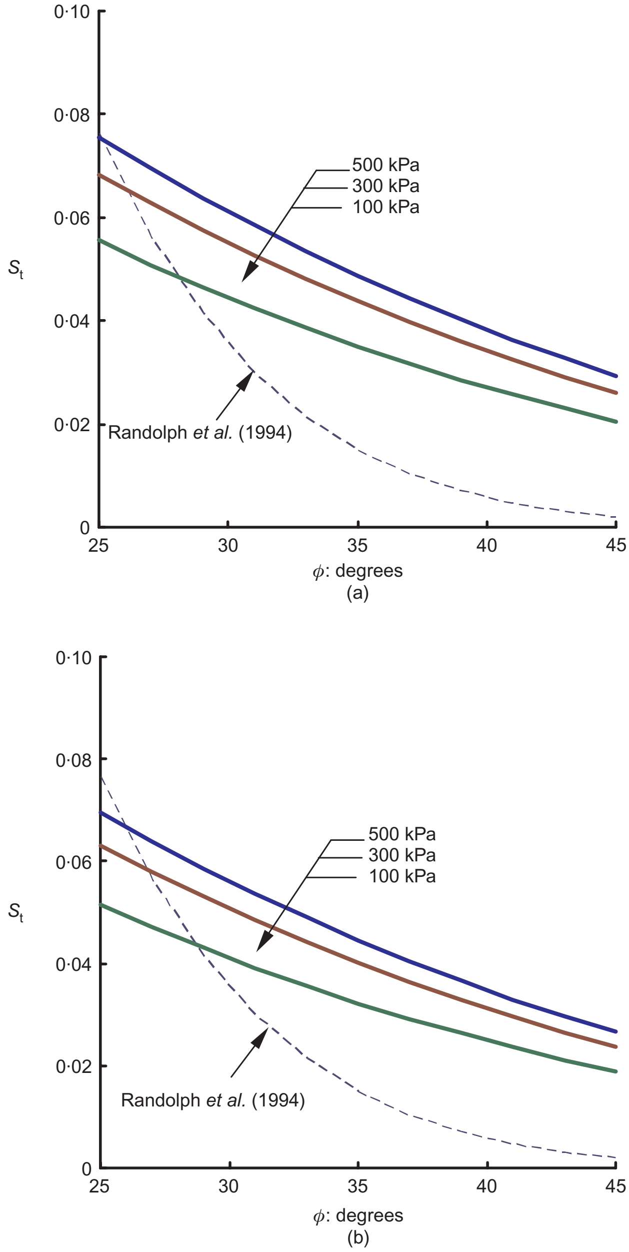

Fig. 3. Predicted S_t values for various states of sand: (a) $D_r = 50\%$; (b) $D_r = 80\%$

and dense sand ($D_r = 80\%$). For each case three mean stress levels (100, 300 and 500 kPa) are taken into consideration, roughly representing stress levels for piles with different embedded lengths. In producing these data, the shear modulus of sand has been estimated using the correlation of Lo Presti (1987)

$$\frac{G}{p_a} = 400 \exp(0{\cdot}7 D_r)\left(\frac{\sigma_0'}{p_a}\right)^{0{\cdot}5} \tag{19}$$

where σ_0' ($\sim p_0'$) is the mean effective confining stress, p_a is a reference pressure (100 kPa), and D_r is relative density. The average volumetric strain, Δ, has been estimated using the empirical correlation of Yasufuku *et al.* (2001)

$$\Delta = 50(I_r)^{-1{\cdot}8} \tag{20}$$

There are several features that are worth noting. First, the value of S_t always decreases with increasing friction angle. This trend is consistent with that predicted using the proposal of Randolph *et al.* (1994), but the new proposal gives a much lower reduction rate. Second, for a given friction angle and relative density, S_t tends to increase with increasing stress level or penetration depth. Third, for a given friction angle and stress level, S_t tends to reduce with increasing relative density.

An alternative comparison of the S_t values from the new and existing proposals is presented in Figs 4 and 5. It is seen that, at a low friction angle ($\phi = 25°$), the proposal of Randolph *et al.* (1994) always gives the highest value (0·08) and the proposal of Fleming *et al.* (1992) gives the smallest (0·02), with the prediction from the new proposal being in between (0·05–0·07). If the friction angle becomes higher ($\phi = 35°$), S_t has values between 0·03 and 0·05, whereas the proposal of Randolph *et al.* (1994) gives a value as low as 0·015.

From a practical point of view, the above comparison may help partly explain why the method of Randolph *et al.*

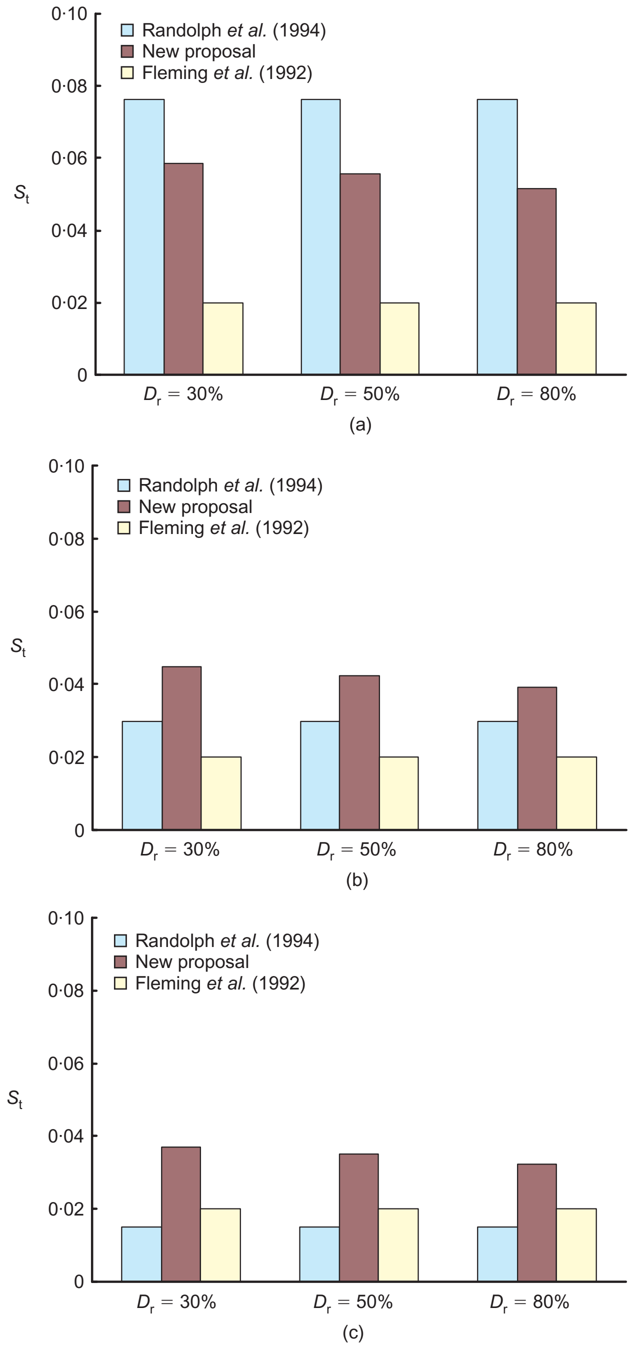

Fig. 4. Values of S_t estimated with different proposals for various states of sand (stress level = 100 kPa): (a) $\phi = 25°$; (b) $\phi = 31°$; (c) $\phi = 35°$

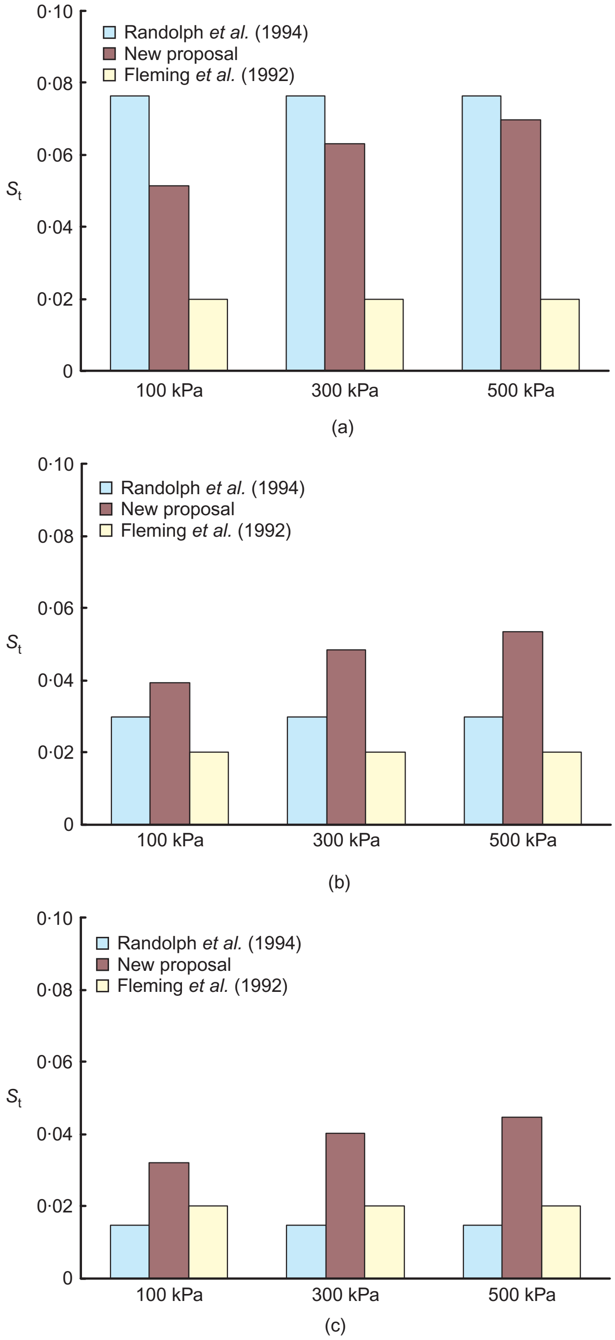

Fig. 5. Values of S_t estimated with different proposals for various states of sand (relative density = 80%): (a) $\phi = 25°$; (b) $\phi = 31°$; (c) $\phi = 35°$

(1994) involving the use of S_t has led to significant underestimates of shaft resistance (see Fig. 6). The largest measured-to-predicted ratio, 2·12, is with a driven pile of 6·7 m (Beringen *et al.*, 1979). While many factors may contribute to this large discrepancy, attention is drawn to the fact that the pile was installed in *very dense, highly overconsolidated* sand with high shearing resistance – recalling that the larger the friction angle, the greater the difference between S_t values from the new and existing proposals.

The correlation given in equation (19) is applicable primarily to clean silica sands containing no or less than 5% fines. If sand has a high fines content, its stiffness may decrease substantially. Yang (2006) has shown that the reduction of stiffness can have a significant impact on the size of the influence zone. Recalling the preliminary analysis in Table 1, it becomes necessary to investigate the impact of stiffness on

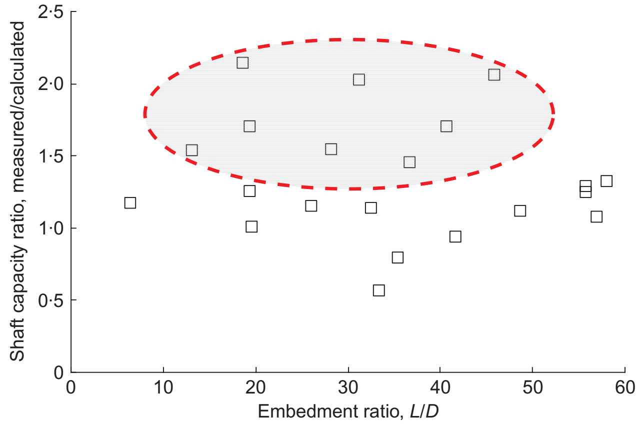

Fig. 6. Measured-to-calculated shaft resistance for a database of pile load tests (after Randolph *et al.*, 1994)

S_t. In doing this, the following correlation for sands containing about 15–30% fines (Lo Presti, 1987) is adopted.

$$\frac{G}{p_a} = 75\exp(0{\cdot}7D_r)\left(\frac{\sigma_0'}{p_a}\right)^{0{\cdot}5} \tag{21}$$

The newly calculated S_t values are presented in Fig. 7, together with the results obtained previously for higher stiffness for comparison. A marked feature is that the reduction of stiffness leads to greater S_t values: for a range of angles (30–35°), S_t can become as large as 0·1. Such a large value was indeed recorded in recent model tests (Gavin & Gallagher, 2005). Another point of interest is that, when the stiffness is reduced, the influence of stress level and friction angle on the value of S_t tends to be more profound.

Since the existing proposal for S_t does not account for the factors of stiffness and stress level, the comparison in Fig. 7 implies that, given otherwise identical input, greater underestimation of shaft resistance might be produced for piles in silty sands than in clean sands. Interestingly, there is a clue in this respect in the case studies reported by Randolph *et al.* (1994): quite large measured-to-predicted values (2·02 and 1·99) were also obtained for two piles driven in a sand deposit containing a significant amount of silt (Mansur & Kaufman, 1956).

Lastly, it should be noted that in the cavity expansion analysis, the shear stiffness affects the rigidity index – an indicator for the average volumetric strain of the sand near the pile tip. In this context, an alternative view of the effect of stiffness is as follows: when the sand becomes more compressible, S_t tends to take larger values, suggesting that care should be taken about possible underestimation of shaft capacity in highly compressible sand.

CONCLUSIONS

The main findings and design implications derived from this study are summarised as follows.

(*a*) The value of S_t is dependent on several factors – the friction angle, shear stiffness, compressibility, relative density and mean stress level of the sand near the pile tip – and these factors are interrelated.

(*b*) Given otherwise identical parameters, S_t tends to decrease with an increase in friction angle, and with a decrease in mean stress level. It also tends to decrease with increasing relative density.

(*c*) For a typical range of parameters, S_t has values varying from 0·03 to 0·05. It may take higher values (~0·1) if the stiffness of sand becomes significantly low.

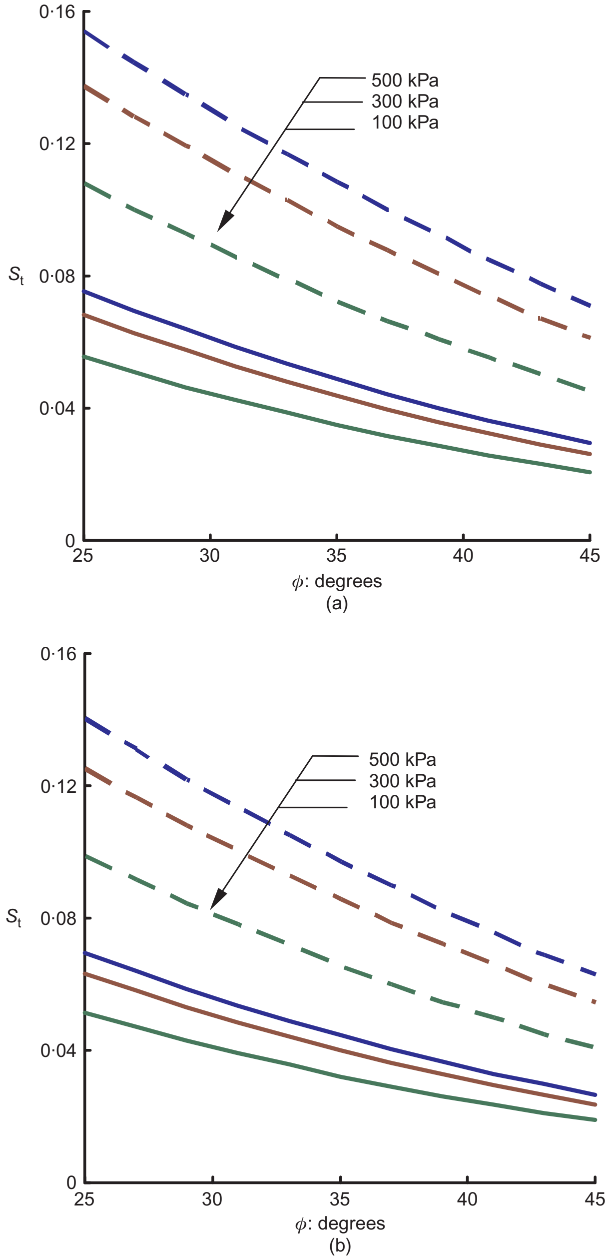

Fig. 7. Effect of stiffness on S_t values (solid lines, high stiffness; broken lines, low stiffness): (a) $D_r = 50\%$; (b) $D_r = 80\%$

(*d*) The existing proposals generally provide lower values for S_t (0·015–0·03), which may contribute to the observed underestimation of shaft resistance, particularly for piles installed in dense sand or in highly compressible sand.

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NOTATION

a, b	empirical parameters
D	pile diameter
D_r	relative density
I_{FU}	upper limit of influence zone
G	shear modulus
I_r	rigidity index
p_a	reference pressure
p_u	cavity pressure
p'_0	mean effective stress at pile tip
q_b	pile base resistance
R_p	plastic zone radius at limit state
R_u	cavity radius at limit state
r	radial distance
S_t	ratio between $\sigma'_{r,max}$ and q_b
β	angle formed by two lines CO and CE (Fig. 2)
Δ	average volumetric strain in plastic zone
δ	interface friction angle between pile and soil
ζ	ratio of R_p to R_u
λ	proportional factor
σ'_r	radial stress component
$\sigma'_{r,max}$	maximum radial effective stress
σ'_{rp}	radial stress in plastic zone
σ'_θ	circumferential stress component
σ'_0	mean effective confining stress
τ_{max}	maximum shaft friction
ϕ	friction angle of sand near pile tip
χ	dimensionless parameter
ψ	angle of soil wedge below pile tip

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