The Relation between SPX Options and the CBOE-listed Volatility Derivatives

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Abstract

We propose a framework, which jointly prices SPX options, volatility index and volatility derivatives, to examine the relation between SPX options and the CBOE-listed volatility derivatives. We obtain analytical formulas for SPX options, VIX, VIX futures, and the first three conditional moments of VIX option. We estimate parameters sequentially by using different data sets. We show that the model can simultaneously capture the dynamics of VIX term structure and the implied volatility of VIX options.

1 Introduction

The advantages of combining the underlying index and its options data to study the dynamics of the index have been emphasized by Chernov and Ghysels (2000), Pan (2002), Jones (2003), and Eraker (2004). Similarly, the joint implied volatility of the index options and volatility derivatives data should be used to investigate the dynamics of volatility. The fast growth of the CBOE-listed volatility derivatives markets not only provides excellent data source on examining the features of volatility, but also poses particular challenge to a unified framework which can price index options and volatility derivatives simultaneously. Although the importance of consistently modeling the index options and volatility derivatives has been recognized in recent years[2], a model which can also capture the dynamics of implied volatility term structure is not yet available. Furthermore, there is no study on the relation between SPX options and the CBOE-listed volatility derivatives markets, which is important for us to improve the understanding of option pricing beyond SPX options market. We fill this gap by providing a theoretical model to jointly price SPX options, the CBOE volatility index (VIX) and VIX derivatives, and a novel methodology to estimate model parameters.

We link SPX options and the CBOE-listed volatility derivatives by using the VIX, which is widely accepted by industry as a benchmark for stock market volatility. Because VIX is a hypothetical implied volatility of SPX options[3], it reveals important information contained in the SPX options. Moreover, the liquid market for VIX futures and options allows us to explore the unique features of volatility from a different angle. Therefore, VIX provides a natural connection between SPX options and VIX derivatives.

In this paper, we study the relation between SPX options and the CBOE-listed volatility markets by considering a two-factor stochastic volatility model for the instantaneous

[2] For example, Buehler (2006), Gatheral (2008) and Cont and Kokholm (2009).
[3] See the CBOE 2003 whitepaper, which is further updated in 2009.

variance, with the second factor to be the long term mean level of the instantaneous variance. We first derive moment properties of the instantaneous variance, which can be used to obtain analytical formulas for VIX, VIX futures, and the first three conditional moments of VIX option. We also calculate European call options on SPX via Fourier inversion. In particular, we model the instantaneous variance to be mean-reverting and its long term mean to be fully persistent. Another important feature of our model is that there is jump in volatility, which is able to produce the positive skew observed in the implied volatilities of VIX options. More importantly, the property of our model enable us to estimate parameters sequentially by using different data sets, which will be discussed in more detail later.

Some related studies are Cont and Kokholm (2009), Mencia and Sentana (2009) and Lin and Chang (2010). Cont and Kokholm (2009) propose a model for the joint dynamics of the underlying index and its variance swaps by directly modeling the variance swaps. However, they only use the data on one day for different maturities to examine the performance of the model. Mencia and Sentana (2009) extend the log-normal Ornstein-Uhlenbeck process of volatility with the emphasis on finding proper generalizations to price VIX derivatives across different market situations. They find that a model with central tendency and stochastic volatility is necessary to price VIX futures and options, respectively. Lin and Chang (2010) introduce a model with correlated jumps in SPX and volatility to investigate VIX option pricing and hedging. Our paper differentiate from these studies by following important features: (1) different from Cont and Kokholm (2009) and Mencia and Sentana (2009), we model the instantaneous volatility of the underlying index, which is in line with previous literature (For example, Heston (1993), Duffie, Pan and Singleton (2000)), (2) we assume the long term mean level of the instantaneous volatility to be the second factor which is constant in Lin and Chang (2010), (3) we derive a simple and intuitive formula for VIX as demonstrated in Luo and Zhang (2009), (4) we make use of different data sets to estimate parameters step by step.

The rest of the paper is organized as follows. Section 2 proposes model for the SPX index and obtain theoretical results. Section 3 describes data sets. Section 4 provides estimation procedure and empirical results. Section 5 concludes the paper.

2 Model

In this section, we describe model setup and derive analytical formulas for SPX option, VIX, VIX futures, and the first three conditional moments of the instantaneous variance and VIX option. The VIX derivatives has been studied in Zhang and Zhu (2006), Zhu and Zhang (2007), Lin (2007), Sepp (2008a, 2008b), Carr and Lee (2009), Lin and Chang (2009), Lu and Zhu (2010), Zhang, Shu, and Brenner (2010), Wang and Daigler (2010). We specify a particular two-factor stochastic volatility model under the framework of Luo and Zhang (2009) with the second factor to be the long term mean level of the instantaneous variance. The necessary of the second stochastic volatility factor has been addressed recently by Adrian and Rosenberg (2008), Christoffersen, Heston and Jacobs (2009), Christoffersen, Jacobs, Ornthanalai and Wang (2008), Egloff, Leippold, and Wu (2010), Luo and Zhang (2009) and Zhang, Shu, and Brenner (2010).

2.1 Model setup

Consider the following model for the SPX index, S_t, under the risk-neutral measure Q,

$$d\ln S_t = \left(r - \frac{1}{2}V_t\right)dt + \sqrt{V_t}\left(\rho dB^Q_{2,t} + \sqrt{1-\rho^2}dB^Q_{1,t}\right), \quad (1)$$

$$dV_t = \kappa(\theta_t - V_t)dt + \sigma_V\sqrt{V_t}dB^Q_{2,t} + ydN_t - \lambda_t E^Q(y)dt, \quad (2)$$

$$d\theta_t = \sigma_\theta dB^Q_{3,t}, \quad (3)$$

where r is the risk-free rate, V_t is the instantaneous variance, $B^Q_{1,t}$, $B^Q_{2,t}$, and $B^Q_{3,t}$ are three independent Brownian motions. ρ is the correlation between index and variance diffusions.

κ is the mean-reverting speed of the instantaneous variance. θ_t is the long-term mean level of V_t. N_t is a Poisson process with intensity λ_t. y is the jump size with mean $\mu_y \equiv E^Q(y)$.

2.2 The moment properties of the instantaneous variance

The moment properties of V_t is described in the following proposition.

Proposition 1 *Under the framework described in Equation (1), the first three conditional (central) moments, M_i, $i = 1, 2, 3$, of the future variance, V_s, $0 < t < s$, can be calculated as follows*

$$\begin{aligned} M_1 &\equiv E_t^Q(V_s) = \theta_t + (V_t - \theta_t)e^{-\kappa(s-t)}, & (4) \\ M_2 &\equiv E_t^Q\left[\left(V_s - E_t^Q(V_s)\right)^2\right] \\ &= \sigma_V^2\left[V_t e^{-\kappa(s-t)}\frac{1-e^{-\kappa(s-t)}}{\kappa} + \theta_t\frac{\left(1-e^{-\kappa(s-t)}\right)^2}{2\kappa}\right] \\ &\quad + e^{-2\kappa s}\int_t^s E_t^Q(\lambda_u y^2)e^{2\kappa u}du, & (5) \\ M_3 &\equiv E_t^Q\left[\left(V_s - E_t^Q(V_s)\right)^3\right] \\ &= \frac{3}{2}\sigma_V^4 V_t e^{-\kappa(s-t)}\frac{\left(1-e^{-\kappa(s-t)}\right)^2}{\kappa^2} + \frac{1}{2}\sigma_V^4\theta_t\frac{\left(1-e^{-\kappa(s-t)}\right)^3}{\kappa^2} \\ &\quad + e^{-3\kappa s}\int_t^s 3\sigma_V^2 e^{\kappa u}\int_t^u E_t^Q(\lambda_v y^2)e^{2\kappa v}dvdu & (6) \\ &\quad + 3e^{-3\kappa s}\int_t^s e^{3\kappa u}E_t^Q(\lambda_u X_u y^2)du + e^{-3\kappa s}\int_t^s e^{3\kappa u}E_t^Q(\lambda_u y^3)du. \end{aligned}$$

where E_t^Q stands for the conditional expectation in the risk-neutral measure.

Proof. See Appendix A.

Particularly, when $\lambda_t = \lambda_0 + \lambda_1 V_t$, and jump size, y, is constant, we have

$$M_2 = \left(\sigma_V^2 + \lambda_1 y^2\right)\left[V_t e^{-\kappa(s-t)}\frac{1-e^{-\kappa(s-t)}}{\kappa} + \theta_t\frac{\left(1-e^{-\kappa(s-t)}\right)^2}{2\kappa}\right] + \lambda_0 y^2\frac{1-e^{-2\kappa(s-t)}}{2\kappa}, \quad (7)$$

and

$$
\begin{aligned}
M_3 \;=\;& \frac{3}{2}\sigma_V^4 V_t e^{-\kappa(s-t)}\frac{\left(1-e^{-\kappa(s-t)}\right)^2}{\kappa^2} + \frac{1}{2}\sigma_V^4\theta_t\frac{\left(1-e^{-\kappa(s-t)}\right)^3}{\kappa^2} \\
&+3\sigma_V^2 y^2\left[(\lambda_0+\lambda_1\theta_t)\frac{1-3e^{-2\kappa(s-t)}+2e^{-3\kappa(s-t)}}{6\kappa^2} + \lambda_1(V_t-\theta_t)\frac{e^{-\kappa(s-t)}(1-e^{-\kappa(s-t)})^2}{2\kappa^2}\right] \\
&+3\lambda_1 y^2(\sigma_V^2+\lambda_1 y^2)\left[V_t e^{-\kappa(s-t)}\frac{(1-e^{-\kappa(s-t)})^2}{2\kappa^2} + \theta_t\frac{(1-e^{-\kappa(s-t)})^3}{6\kappa^2}\right] \\
&+\lambda_0\lambda_1 y^4\frac{1-3e^{-2\kappa(s-t)}+2e^{-3\kappa(s-t)}}{2\kappa^2} + \lambda_0 y^3\frac{1-e^{-3\kappa(s-t)}}{3\kappa} \\
&+\lambda_1 y^3\left[\theta_t\frac{1-e^{-3\kappa(s-t)}}{3\kappa} + (V_t-\theta_t)\frac{e^{-\kappa(s-t)}-e^{-3\kappa(s-t)}}{2\kappa}\right]. \qquad (8)
\end{aligned}
$$

Furthermore, if we assume $\lambda_1 = 0$, then

$$
M_2 \;=\; \sigma_V^2\left[V_t e^{-\kappa(s-t)}\frac{1-e^{-\kappa(s-t)}}{\kappa} + \theta_t\frac{\left(1-e^{-\kappa(s-t)}\right)^2}{2\kappa}\right] + \lambda_0 y^2\frac{1-e^{-2\kappa(s-t)}}{2\kappa}, \qquad (9)
$$

and

$$
\begin{aligned}
M_3 \;=\;& \frac{3}{2}\sigma_V^4 V_t e^{-\kappa(s-t)}\frac{\left(1-e^{-\kappa(s-t)}\right)^2}{\kappa^2} + \frac{1}{2}\sigma_V^4\theta_t\frac{\left(1-e^{-\kappa(s-t)}\right)^3}{\kappa^2} \\
&+\sigma_V^2 y^2\lambda_0\frac{1-3e^{-2\kappa(s-t)}+2e^{-3\kappa(s-t)}}{2\kappa^2} + \lambda_0 y^3\frac{1-e^{-3\kappa(s-t)}}{3\kappa}, \qquad (10)
\end{aligned}
$$

which will be used in our estimation.

2.3 VIX and its moment properties

In this subsection, we derive expressions for VIX, VIX futures and the moment properties of log VIX.

2.3.1 VIX and VIX futures formulas

Since the innovation terms in the dynamics of V_t and θ_t are martingale processes, as demonstrated in Luo and Zhang (2009), the VIX formula can be easily obtained as a simple function of V_t and θ_t. The result is presented in the following proposition.

Proposition 2 *Under the framework described in Equation (1), the VIX index squared, at time t, with maturity τ, $VIX^2_{t,\tau}$, can be obtained as*

$$VIX^2_{t,\tau} = (1-\alpha)\theta_t + \alpha V_t, \quad \alpha = \frac{1-e^{-k\tau}}{k\tau}. \tag{11}$$

Proposition 3 *For a general function of θ_T and V_T, $f(\theta_T, V_T)$, approximation of its expectation near the point of $(\theta_t, E^Q_t(V_T))$, up to $O(\sigma^6_V)$, is given by*

$$\begin{aligned} E^Q_t\left[f(\theta_T, V_T)\right] &= f(\theta_t, E^Q_t(V_T)) + \frac{1}{2} f_{\theta\theta}(\theta_t, E^Q_t(V_T))\sigma^2_\theta(T-t) \\ &\quad + \frac{1}{2} f_{VV}(\theta_t, E^Q_t(V_T))M_2(T) + \frac{1}{6} f_{VVV}(\theta_t, E^Q_t(V_T))M_3(T). \end{aligned} \tag{12}$$

Proof. See Appendix B.

Because $VIX_{T,\tau} = \sqrt{(1-\alpha)\theta_T + \alpha V_T}$, an approximate formula for the VIX futures price can be obtained by applying the result in the Proposition 3. The formula is presented in the following corollary.

Corollary 1: *The price of VIX futures with maturity T, F^T_t, is given by*

$$\begin{aligned} \frac{F^T_t}{100} &= E^Q_t\left[VIX_{T,\tau}\right] \\ &= \left[\theta_t(1-\alpha e^{-\kappa(T-t)}) + V_t \alpha e^{-\kappa(T-t)}\right]^{\frac{1}{2}} \\ &\quad - \frac{1}{8}\left[\theta_t(1-\alpha e^{-\kappa(T-t)}) + V_t \alpha e^{-\kappa(T-t)}\right]^{-\frac{3}{2}} \alpha^2 M_2 \\ &\quad + \frac{1}{16}\left[\theta_t(1-\alpha e^{-\kappa(T-t)}) + V_t \alpha e^{-\kappa(T-t)}\right]^{-\frac{5}{2}} \alpha^3 M_3 \\ &\quad - \frac{1}{8}(1-\alpha)^2\sigma^2_\theta(T-t)\left[(1-\alpha)\theta_t + \alpha E^Q_t(V_T)\right]^{-\frac{3}{2}}, \end{aligned} \tag{13}$$

where terms with order $O(\sigma^6_V)$ have been ignored, and we assume $O(\sigma_\theta) \sim O(\sigma^2_V)$.

2.3.2 The moment properties of log VIX

Let $Y_{t,T} = \ln \frac{VIX_{T,\tau}}{VIX_{t,\tau}}$ and $U_T = \ln\left[(1-\alpha)\theta_T + \alpha V_T\right]$, then, the moment properties of $Y_{t,T}$ can be obtained as following

Proposition 4 *The first three conditional (central) moments, $\bar{M}_i$, $i = 1, 2, 3$, of $Y_{t,T}$ can be calculated as follows*

$$\begin{aligned}
\bar{M}_1 &\equiv E_t^Q\left(Y_{t,T}\right), \\
&= \frac{1}{2}\left[\ln b - \ln\left[(1-\alpha)\theta_t + \alpha V_t\right] - \frac{1}{2}\left(\frac{\alpha}{b}\right)^2 M_2 + \frac{1}{3}\left(\frac{\alpha}{b}\right)^3 M_3 - \frac{1}{2}\left(\frac{1-\alpha}{b}\right)^2 \sigma_\theta^2 (T-t)\right],
\end{aligned} \tag{14}$$

$$\begin{aligned}
\bar{M}_2 &\equiv E_t^Q\left[\left(Y_{t,T} - E_t^Q(Y_{t,T})\right)^2\right], \\
&= \frac{1}{4}\left[\left(\frac{1-\alpha}{b}\right)^2 \sigma_\theta^2 (T-t) + \left(\frac{\alpha}{b}\right)^2 M_2 - \left(\frac{\alpha}{b}\right)^3 M_3\right],
\end{aligned} \tag{15}$$

$$\begin{aligned}
\bar{M}_3 &\equiv E_t^Q\left[\left(Y_{t,T} - E_t^Q(Y_{t,T})\right)^3\right], \\
&= \frac{1}{8}\left[E_t^Q(U_T^3) - 3E_t^Q(U_T)E_t^Q(U_T^2) + 2\left(E_t^Q(U_T)\right)^3\right],
\end{aligned} \tag{16}$$

where $b \equiv (1-\alpha)\theta_t + \alpha E_t^Q(V_T)$, $E_t^Q(U_T^3)$, $E_t^Q(U_T^2)$ and $E_t^Q(U_T)$ are given in Appendix C.

Proof. See Appendix C.

2.4 VIX option

In this subsection, we use the results in Zhang and Xiang (2008) to obtain VIX option data implied the second and third moments, which will be used in later estimation.

We suppose the VIX in the risk-neutral world is given by

$$VIX_{T,\tau} = VIX_{t,\tau} e^{(-\frac{1}{2}\sigma^2+\mu)(T-t)+\sigma\sqrt{T-t}z}, \tag{17}$$

where μ is the convexity adjustment, z is a random number with mean zero, variance 1, skewness ν. Because $Y_{t,T} = \ln\frac{VIX_{T,\tau}}{VIX_{t,\tau}}$, we have

$$Y_{t,T} = \left(-\frac{1}{2}\sigma^2 + \mu\right)(T-t) + \sigma\sqrt{T-t}z \Longrightarrow \mu = \frac{\bar{M}_1}{T-t} + \frac{1}{2}\sigma^2, \tag{18}$$

$$Var(Y_{t,T}) = \sigma^2 (T-t) Var(z) \Longrightarrow \sigma^2 = \frac{\bar{M}_2}{T-t}, \tag{19}$$

$$Skew(Y_{t,T}) = (\sigma\sqrt{T-t})^3 Skew(z) \Longrightarrow \nu = \frac{\bar{M}_3}{(\sigma\sqrt{T-t})^3}. \tag{20}$$

Furthermore, the relation between the level and slope of the VIX option implied volatility smirk, (γ_0, γ_1), and the risk-neutral standard deviation and skewness, (σ, ν) are given by

$$\gamma_0 = \left(1 - \frac{\lambda_2}{24}\right)\sigma \approx \sigma, \tag{21}$$

$$\gamma_1 = \frac{1}{6}\nu, \tag{22}$$

where λ_2 is the curvature of the smirk and $\lambda_2/24 \ll 1$. Using equations in (19) and (20), we have the second and third moments implied by VIX options data

$$\hat{M}_2 = \gamma_0^2(T-t), \tag{23}$$

$$\hat{M}_3 = 6\gamma_1(\gamma_0\sqrt{T-t})^3, \tag{24}$$

which can be used to estimate parameters.

2.5 SPX option

In this subsection, we derive analytical formula for European call options on SPX via Fourier inversion. We consider two special cases by specifying the jump intensity to be constant and random, respectively.

For the index option pricing, the key is to derive the moment generating function of the future index $s_T \equiv \ln S_T$ conditional upon current observing variable $s_t \equiv \ln S_t$. Let

$$f(\phi; s_t, t, T) \equiv E_t^Q[e^{\phi s_T}|\mathcal{F}_t], \tag{25}$$

be the moment generating function of the s_T. Since f is a martingale, it satisfies backwards Kolmogorov equation

$$\begin{aligned} -f_\tau + \left(r - \frac{1}{2}V_t\right)f_s + \frac{1}{2}V_t f_{ss} &+ [\kappa(\theta_t - V_t) - \lambda_t\mu_y]f_V + \frac{1}{2}\sigma_V^2 V_t f_{VV} + \rho\sigma_V V_t f_{sV} \\ &+ \frac{1}{2}\sigma_\theta^2 f_{\theta\theta} + \lambda_t E[f(V_t + y) - f(V_t)] = 0, \end{aligned} \tag{26}$$

with a boundary condition $f(\phi; s_T, t+\tau, 0) = e^{\phi s_T}$, and $\tau \equiv T - t$. For the affine jump diffusion processes considered here, the solution is exponentially affine in the state variables with the following form

$$f(\phi; s_t, \tau) = e^{A(\phi;\tau)+B(\phi;\tau)V_t+C(\phi;\tau)\theta_t+\phi s_t}, \tag{27}$$

and the boundary conditions $A(0) = B(0) = C(0) = 0$. Substituting the guess solution into the above Kolmogorov equation, we have

$$\begin{aligned} &-\left[A^{'}(\tau) + B^{'}(\tau)V_t + C^{'}(\tau)\theta_t\right] + \left(r - \frac{1}{2}V_t\right)\phi + \frac{1}{2}V_t\phi^2 + [\kappa(\theta_t - V_t) - \lambda_t\mu_y]B(\tau) \\ &\quad + \frac{1}{2}\sigma_V^2 B^2(\tau)V_t + \rho\sigma_V\phi B(\tau)V_t + \frac{1}{2}\sigma_\theta^2 C^2(\tau) + \lambda_t E[e^{B(\tau)y} - 1] = 0. \end{aligned} \tag{28}$$

The solutions of $A(\tau)$, $B(\tau)$ and $C(\tau)$ depends on the specification of λ_t. We consider two special cases in the following.

2.5.1 Constant jump intensity case

When $\lambda_t = \lambda_0$, we have the following ODEs

$$A^{'}(\tau) = r\phi - \lambda_0\mu_y B(\tau) + \frac{1}{2}\sigma_\theta^2 C^2(\tau) + \lambda_0 E[e^{B(\tau)y} - 1], \tag{29}$$

$$B^{'}(\tau) = \frac{1}{2}\phi(\phi - 1) + (\phi\rho\sigma_V - \kappa)B(\tau) + \frac{1}{2}\sigma_V^2 B^2(\tau), \tag{30}$$

$$C^{'}(\tau) = \kappa B(\tau). \tag{31}$$

$B(\tau)$ can be solved explicitly from the Riccati equation,

$$B(\phi; \tau) = \frac{\phi(\phi - 1)(1 - e^{-\zeta\tau})}{2\zeta - \vartheta(1 - e^{-\zeta\tau})}, \tag{32}$$

where

$$\zeta \equiv \sqrt{(\phi\rho\sigma_V - \kappa)^2 - \phi(\phi - 1)\sigma_V^2}, \qquad \vartheta \equiv \zeta + \phi\rho\sigma_V - \kappa \tag{33}$$

Then, we have

$$C(\tau) = -\frac{\kappa}{\sigma_V^2}\vartheta\tau - \frac{2\kappa}{\sigma_V^2}\frac{\vartheta - \zeta}{\zeta}\ln\left(1 - \frac{\vartheta(1 - e^{-\zeta\tau})}{2\zeta}\right), \tag{34}$$

and

$$A(\tau) = r\phi\tau - \frac{\lambda_0\mu_y}{\kappa}C(\tau) + \frac{1}{2}\sigma_\theta^2\int_0^\tau C^2(u)du + \lambda_0\left[\int_0^\tau E[e^{B(u)y}]du - \tau\right]. \quad (35)$$

Therefore, the characteristic function of $\ln(S_T/S_t)$ is given by $g(\phi;\tau) = e^{-i\phi s_t}f(i\phi)$, and European call options, with strike price K and maturity τ, are given by

$$c(t,\tau) = S_t\Pi_1 - Ke^{-r\tau}\Pi_2, \quad (36)$$

where

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln(K/S)}g(\phi - i;\tau)}{i\phi g(-i;\tau)}\right]d\phi, \quad (37)$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln(K/S)}g(\phi;\tau)}{i\phi}\right]d\phi. \quad (38)$$

2.5.2 Random jump intensity case

When $\lambda_t = \lambda_1 V_t$, we have the following ODEs

$$A'(\tau) = r\phi + \frac{1}{2}\sigma_\theta^2 C^2(\tau), \quad (39)$$

$$B'(\tau) = \frac{1}{2}\phi(\phi-1) + (\phi\rho\sigma_V - \kappa - \lambda_1\mu_y)B(\tau) + \frac{1}{2}\sigma_V^2 B^2(\tau) + \lambda_1 E[e^{B(\tau)y} - 1], \quad (40)$$

$$C'(\tau) = \kappa B(\tau). \quad (41)$$

In this case, numerical method should be used to calculate $A(\tau)$, $B(\tau)$ and $C(\tau)$.

3 Data

We use the daily VIX options market data, from March 22, 2006 to June 8, 2007, to calculate VIX options implied volatility and skewness. The daily VIX term structure data which match the sample period of VIX options are constructed by Luo and Zhang (2009).

3.1 VIX options data

The daily VIX options market data employed in this paper are from March 22, 2006 to June 8, 2007. Due to zero prices for VIX options on September 6, 2006 and November 10, 2006, we drop data on these two days and have totally 304 observations. We also use U.S. treasury daily yield curve rates during the same period above, provided by the U.S. Treasury Department. We construct risk-free interest rates for difference maturities on each trading day by using linear interpolation. The VIX options data contains trading date and expiration data; type of options (call or put) and strike price; open, high, low and close prices; trading volume and open interests; last bid and last ask prices; underlying VIX index values.

We notice that the expiration dates in the original VIX options data set are not the true expiration dates. They are the third or fourth Saturdays of the months, while the VIX options expiration dates are Wednesdays. Therefore we need to transfer the original expiration dates to true expiration dates before we start to process the data. For example, if the original expiration dates are March 18, 2006 and April 22, 2006, we first find the third Friday in the next month of the original expiration dates, which are April 21, 2006 and May 19, 2006. We then subtract 30 days from these "third Fridays in the next month" and obtain some Wednesdays in the original months, and these Wednesdays, such as March 22, 2006 and April 19, 2006 for the example, are the true expiration dates. We obtain the days to maturity by calculating the number of calender days between the trading date and the true expiration date, and we then divide it by 365 to annualize the time to maturity. For example, if the trading date is March 1, 2006, and true expiration dates are March 22, 2006 and April 19, 2006, then the days to maturity are 21 days and 49 days, and annualized time to maturity are 0.0575 and 0.1342 year, respectively. The U.S. treasury daily yield curve rates include treasury yields for 1 month, 3 months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, 20 years, and 30 years, on each trading day. We find risk-free interest rates for different expirations by using linear interpolation with the available daily treasury yield curve rates.

Now, we introduce the methodology to calculate the ATM implied volatility and its slope, which will be used in the calibration. At first, the implied forward price is obtained from the options market by using put-call parity, i.e., $c_t(K) - p_t(K) = S_t - Ke^{-r(T-t)}$. The forward price at time t with maturity T is given by $F_{implied}(K) = S_t e^{r(T-t)} = [c_t(K) - p_t(K)]e^{r(T-t)} + K$. Notice that the implied forward price could be a function of strike price K. On each trading day, for the options with the same expiration date, we list the calls and puts separately according to their strike prices. We define the pair of call and put options with the same strike and the smallest absolute value of the price difference as the *ATM options*. The implied forward price computed from the pair of ATM call and put is called *ATM implied forward price*.

The *implied volatility* of an option is defined as the volatility that equates the Black-Scholes option price with the market price of the option, i.e.,

$$c_t^{market}(K, T-t) = c_t^{Black-Scholes}(\sigma; K, T-t) = F_t^T e^{-r(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2), \quad (42)$$

where F_t^T is the ATM implied forward price, and

$$d_1 = \frac{\ln \frac{F_t^T}{K} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

The *ATM implied volatility* is regarded as the implied volatility for the option with strike price being equal to the ATM implied forward price, i.e., $K = F_t^T$. The put option with strike price lower than the ATM implied forward price is called *out of money (OTM) put*. The call option with strike price higher than the ATM implied forward price is called *OTM call*. Using the OTM puts and calls, we can construct the implied volatility as a function of strike price for the VIX options with the same maturity. We look for two strike prices,

K_{lower} and K_{upper}, that are the nearest strike below and above the ATM implied forward price, and then calculate the ATM implied volatility by using linear interpolation with the implied volatility at these two strike prices.

The *implied volatility skew* is defined as the ATM slope of the implied volatility as a function of strike price, that is

$$skew = \frac{\sigma_{implied}(K_{upper}) - \sigma_{implied}(K_{lower})}{K_{upper} - K_{lower}}. \tag{43}$$

On each trading day, we have computed ATM implied volatilities and implied volatility skews for a set of fixed time-to-maturity, such as 2, 3, 4, 5, and 6 months.

Table 1 provides descriptive statistics for daily VIX option implied ATM volatilities with maturities 2, 3, 4, 5, and 6 months. It can be seen that the average implied volatilities are decrease as maturity increases with 0.711 and 0.554 for 2-month and 6-month implied volatilities, respectively. Figure 1 shows time series of the daily VIX option implied at the money volatilities with selected maturities. The term structure of VIX option implied volatility is downward sloping during most time in the sample period except in April and May in 2006.

3.2 VIX term structure data

We use the daily VIX term structure data constructed by Luo and Zhang (2009) from March 22, 2006 to June 8, 2007, which corresponds to the sample period of VIX options. The maturities are 2, 3, 4, 5, and 6 months.

Following the CBOE's definition, a VIX at time t, with maturity τ, is defined as

$$\begin{aligned} VIX_t(\tau) &= 100 \times \sigma_t(\tau), \\ \sigma_t^2(\tau) &= \frac{2}{\tau} \sum_i \frac{\Delta K_i}{K_i^2} e^{rT_{\text{Calendar}}} Q(K_i) - \frac{1}{\tau} \left[\frac{F}{K_0} - 1 \right]^2, \end{aligned} \tag{44}$$

where T_{Calendar} is a calender day measure that is used to discount the option prices. K_i is the strike price of i-th out-of-money options, ΔK_i is the interval between two strikes, defined as

$\Delta K_i = (K_{i+1} - K_i)/2$. In particular, ΔK_i is the difference between the lowest and the next lowest strikes for the lowest strike and is the difference between the highest and the next highest strikes for the highest strike. r is the time-t risk-free rate to expiration. $Q(K_i)$ is the midpoint of the bid-ask spread of each option with strike K_i. F is the implied forward index level derived from the nearest to the money index option prices by using put-call parity and K_0 is the first strike that is below the forward index level. The calculation uses only out-of-the-money options except at K_0, where $Q(K_0)$ is the average of the call and put option prices at this strike. Note that the CBOE calculate VIX term structure data using a "business day" convention to measure time to expiration, as well as the "calendar day" convention used in the VIX index itself.

Table 2 provides descriptive statistics for the daily VIX term structure data with maturities 2, 3, 4, 5, and 6 months from March 22, 2006 to June 8, 2007. Note that the average VIXs are upward sloping as maturity increases. Figure 2 shows time series of the daily VIX term structure data with selected maturities. The term structure of VIX is upward sloping from March to May in 2006 and August 2006 to February 2007. The term structure is flat in other periods.

4 Estimation

In this section, we show how to estimate different parameters sequentially by using appropriate data sets.

The estimation consists of three phases: In the Phase I, we use the methodology employed in Luo and Zhang (2009) to estimate parameter set $\{\kappa, V_t, \theta_t\}_{t=1,\ldots,T}$ in the Equation (11) by using VIX term structure data. In the Phase II, we match the model implied and VIX option data implied second moment of log VIX to estimate parameter set $\{\lambda_0, \sigma_V, y, \sigma_\theta\}$ by using VIX options data. In the Phase III, we use SPX options data to estimate $\{\rho\}$.

Phase I: we use above daily VIX term structure data to estimate parameters in the VIX formula. Since the stochastic volatility is unobservable, we have to estimate model's parameter, κ, as well as the spot variances $\{V_t\}_{t=1,...,T}$ and its long term mean $\{\theta_t\}_{t=1,...,T}$, where T is the number of observations. We adopt an efficient iterative two-step procedure in Luo and Zhang (2009). The procedure starts from an initial value for κ.

Step 1: Obtain time series of $\{V_t, \theta_t\}$, $t = 1, ..., T$. In particular, for a given parameter set $\{\kappa\}$, we solve T optimization problems of the form:

$$\{\hat{V}_t, \hat{\theta}_t\} = \arg\min \sum_{j=1}^{N_t} \left(VIX_{t,\tau_j}^{Mkt} - VIX_{t,\tau_j} \right)^2, \quad t = 1, ..., T, \tag{45}$$

where VIX_{t,τ_j}^{Mkt} is the market value of VIX with maturity τ_j on day t and VIX_{t,τ_j} is the corresponding theoretical value given by Equation (11). N_t is the number of maturities used on day t.

Step 2: Estimate parameter set $\{\kappa\}$ with $\{V_t, \theta_t\}_{t=1,...,T}$ obtained in Step 1. That is, we minimize aggregate sum of squared errors

$$\{\hat{\kappa}\} = \arg\min \sum_{t=1}^{T} \sum_{j=1}^{N_t} \left(VIX_{t,\tau_j}^{Mkt} - VIX_{t,\tau_j} \right)^2. \tag{46}$$

Phase II: We match the model implied and VIX option data implied second moments of log VIX. The model and data implied second moments are given by (15) and (23), respectively. We estimate parameter set $\{\lambda_0, \sigma_V, y, \sigma_\theta\}$ by minimizing aggregate sum of squared errors

$$\{\hat{\lambda_0}, \hat{\sigma_V}, \hat{y}, \hat{\sigma_\theta}\} = \arg\min \sum_{t=1}^{T} \sum_{j=1}^{N_t} \left(\hat{M}_2(t, \tau_j) - \bar{M}_2(t, \tau_j) \right)^2, \tag{47}$$

where $\bar{M}_2$ and $\hat{M}_2$ are model and data implied second moments, respectively.

Phase III: We use SPX options data to estimate parameter $\{\rho\}$ by minimizing aggregate sum of squared errors

$$\{\hat{\rho}\} = \arg\min \sum_{t=1}^{T} \sum_{j=1}^{N_t^c} \left(\hat{c}(t, \tau_j) - c(t, \tau_j) \right)^2, \tag{48}$$

where $\hat{c}$ is SPX options market data and c is its model price given by Equation (36). N_t^c is the maturity number of SPX options on each day.

Figure 3 shows daily estimates of V_t and θ_t. Note that the long term mean level moves among a relative small range between 0.018 and 0.032 while the instantaneous variance variate substantially with the values from 0.004 to 0.055. It is clear that there are also many large jumps in the instantaneous variance, which means that our model is able to generate sufficient jumps in the volatility. We obtain $\kappa = 7.494$, $\lambda_0 = 0.044$, $\sigma_V = 0.450$, $y = 0.019$, and $\sigma_\theta = 0.035$.

5 Conclusion

In this paper, we propose a consistent model to study the relation between SPX options and the CBOE-listed volatility derivatives marekts. We obtain analytical formulas for SPX options, VIX, VIX futures, and the first three moments of VIX option. We estimate model parameters sequentially by using appropriate data sets. It turns out that the model is able to capture the dynamics of VIX term structure and the implied volatility of VIX options.

A Proof of Proposition 1

We first prove the result for the first moment. From Equation (2), we have

$$E_t^Q(dV_s) = \kappa\left[E_t^Q(\theta_s) - E_t^Q(V_s)\right]ds. \tag{49}$$

Since $E_t^Q(dV_s) = d\left(E_t^Q(V_s)\right)$ and $E_t^Q(\theta_s) = \theta_t$, we obtain an ODE for $E_t^Q(V_s)$

$$d\left(E_t^Q(V_s)\right) = \kappa\left[\theta_t - E_t^Q(V_s)\right]ds, \tag{50}$$

which can be easily solved to get the results in Proposition 1.

Now, we proceed to the second moment. For simplicity, we let $X_s \equiv V_s - E_t^Q(V_s)$. Subtracting Equation (2) with Equation (50), we get

$$\begin{aligned} dX_s &= \kappa(\theta_s - \theta_t - X_s)ds + \sigma_V\sqrt{V_s}dB_{2,s}^Q + ydN_s - \lambda_s E^Q(y)ds \\ &= \left[\kappa(\theta_s - \theta_t - X_s) - \lambda_s E^Q(y)\right]ds + \sigma_V\sqrt{V_s}dB_{2,s}^Q + ydN_s. \end{aligned} \tag{51}$$

By Ito's lemma with jump, we have

$$\begin{aligned} dX_s^2 &= 2X_s\left[\left(\kappa(\theta_s - \theta_t - X_s) - \lambda_s E^Q(y)\right)ds + \sigma_V\sqrt{V_s}dB_{2,s}^Q\right] \\ &\quad + \sigma_V^2 V_s ds + \left[(X_s + y)^2 - X_s^2\right]dN_s, \\ &= \left[2\kappa X_s(\theta_s - \theta_t - X_s) - 2\lambda_s X_s E^Q(y) + \sigma_V^2 V_s\right]ds \\ &\quad + 2\sigma_V X_s\sqrt{V_s}dB_{2,s}^Q + (2yX_s + y^2)dN_s. \end{aligned} \tag{52}$$

Taking conditional expectation on both sides and using $E_t^Q(\theta_s) = \theta_t$, we get

$$\begin{aligned} E_t^Q\left(dX_s^2\right) &= \left[-2\kappa E_t^Q(X_s^2) - 2E_t^Q(\lambda_s X_s)E^Q(y) + \sigma_V^2 E_t^Q(V_s)\right]ds \\ &\quad + E_t^Q\left[(2yX_s + y^2)\lambda_s\right]ds, \\ &= \left[-2\kappa E_t^Q(X_s^2) + \sigma_V^2 E_t^Q(V_s) + E_t^Q(\lambda_s y^2)\right]ds. \end{aligned} \tag{53}$$

Since $E_t^Q(dX_s^2) = d\left(E_t^Q(X_s^2)\right)$, we obtain an ODE for $E_t^Q(X_s^2)$

$$d\left(E_t^Q(X_s^2)\right) = \left[-2\kappa E_t^Q(X_s^2) + \sigma_V^2 E_t^Q(V_s) + E_t^Q(\lambda_s y^2)\right]ds. \tag{54}$$

Let $Y_s = E_t^Q(X_s^2)$ and use $E_t^Q(V_s) = \theta_t + (V_t - \theta_t)e^{-\kappa(s-t)}$, we have

$$\begin{aligned} d\left(e^{2\kappa s}Y_s\right) &= 2\kappa e^{2\kappa s}Y_s ds + e^{2\kappa s}dY_s, \\ &= e^{2\kappa s}\left[\sigma_V^2 E_t^Q(V_s) + E_t^Q(\lambda_s y^2)\right] ds, && (55) \\ &= e^{2\kappa s}\left[\sigma_V^2\left(\theta_t + (V_t-\theta_t)e^{-\kappa(s-t)}\right) + E_t^Q(\lambda_s y^2)\right] ds && (56) \end{aligned}$$

Taking integration on both sides, we get

$$e^{2\kappa s}Y_s = \sigma_V^2\theta_t\frac{e^{2\kappa s}-e^{2\kappa t}}{2\kappa} + \sigma_V^2(V_t-\theta_t)\frac{e^{\kappa(s+t)}-e^{2\kappa t}}{\kappa} + \int_t^s E_t^Q(\lambda_u y^2)e^{2\kappa u}du. \quad (57)$$

Therefore,

$$Y_s = \sigma_V^2 V_t\frac{e^{-\kappa(s-t)}-e^{-2\kappa(s-t)}}{\kappa} + \sigma_V^2\theta_t\frac{(1-e^{-\kappa(s-t)})^2}{2\kappa} + \int_t^s E_t^Q(\lambda_u y^2)e^{-2\kappa(s-u)}du.$$

which is the second moment provided in Proposition 1.

For the third moment, we use similar procedure and have

$$\begin{aligned} dX_s^3 &= 3X_s^2\left[\left(\kappa(\theta_s-\theta_t-X_s) - \lambda_s E^Q(y)\right)ds + \sigma_V\sqrt{V_s}dB_{2,s}^Q\right] \\ &\quad +3X_s\sigma_V^2V_s ds + \left[(X_s+y)^3 - X_s^3\right]dN_s, \\ &= \left[3\kappa X_s^2(\theta_s-\theta_t-X_s) - 3\lambda_s X_s^2E^Q(y) + 3\sigma_V^2X_sV_s\right]ds \\ &\quad +3\sigma_V X_s^2\sqrt{V_s}dB_{2,s}^Q + (3X_s^2y + 3X_sy^2 + y^3)dN_s. && (58) \end{aligned}$$

Then,

$$\begin{aligned} E_t^Q\left(dX_s^3\right) &= \left[-3\kappa E_t^Q(X_s^3) - 3E_t^Q(\lambda_sX_s^2)E^Q(y) + 3\sigma_V^2E_t^Q(X_sV_s)\right]ds \\ &\quad +E_t^Q\left[(3X_s^2y + 3X_sy^2 + y^3)\lambda_s\right]ds, \\ &= \left[-3\kappa E_t^Q(X_s^3) + 3\sigma_V^2E_t^Q(X_sV_s) + 3E_t^Q(\lambda_sX_sy^2) + E_t^Q(\lambda_sy^3)\right]ds, \end{aligned}$$

and the ODE for $E_t^Q(X_s^3)$ is given by

$$d\left(E_t^Q(X_s^3)\right) = \left[-3\kappa E_t^Q(X_s^3) + 3\sigma_V^2E_t^Q(X_s^2) + 3E_t^Q(\lambda_sX_sy^2) + E_t^Q(\lambda_sy^3)\right]ds,$$

where we have used $E_t^Q(X_sV_s) = E_t^Q(X_s^2)$. Let $Z_s = E_t^Q(X_s^3)$ and use the results for the second moment, we have

$$\begin{aligned} d\left(e^{3\kappa s}Z_s\right) &= 3\kappa e^{3\kappa s}Z_s ds + e^{3\kappa s}dZ_s, \\ &= e^{3\kappa s}\left[3\sigma_V^2 E_t^Q(X_s^2) + 3E_t^Q(\lambda_s X_s y^2) + E_t^Q(\lambda_s y^3)\right]ds, \\ &= e^{3\kappa s}\left[3\sigma_V^2 Y_s + 3E_t^Q(\lambda_s X_s y^2) + E_t^Q(\lambda_s y^3)\right]ds, \quad (59) \\ &= 3\sigma_V^2 e^{3\kappa s}\Big[\sigma_V^2 V_t \frac{e^{-\kappa(s-t)} - e^{-2\kappa(s-t)}}{\kappa} + \sigma_V^2\theta_t \frac{(1-e^{-\kappa(s-t)})^2}{2\kappa} \\ &\quad + \int_t^s E_t^Q(\lambda_u y^2)e^{-2\kappa(s-u)}du\Big]ds \\ &\quad + e^{3\kappa s}\left[3E_t^Q(\lambda_s X_s y^2) + E_t^Q(\lambda_s y^3)\right]ds. \quad (60) \end{aligned}$$

Taking integration on both sides, we have

$$\begin{aligned} Z_s &= 3\sigma_V^4 \frac{V_t}{\kappa}\left(\frac{e^{-\kappa(s-t)} - e^{-3\kappa(s-t)}}{2\kappa} - \frac{e^{-2\kappa(s-t)} - e^{-3\kappa(s-t)}}{\kappa}\right) \\ &\quad + 3\sigma_V^4 \frac{\theta_t}{2\kappa}\left(\frac{1-e^{-3\kappa(s-t)}}{3\kappa} - \frac{e^{-\kappa(s-t)} - e^{-3\kappa(s-t)}}{\kappa} + \frac{e^{-2\kappa(s-t)} - e^{-3\kappa(s-t)}}{\kappa}\right) \\ &\quad + e^{-3\kappa s}\int_t^s 3\sigma_V^2 e^{\kappa u}\int_t^u E_t^Q(\lambda_v y^2)e^{2\kappa v}dvdu \\ &\quad + 3e^{-3\kappa s}\int_t^s e^{3\kappa u}E_t^Q(\lambda_u X_u y^2)du + e^{-3\kappa s}\int_t^s e^{3\kappa u}E_t^Q(\lambda_u y^3)du. \\ &= \frac{3}{2}\sigma_V^4 V_t e^{-\kappa(s-t)}\frac{\left(1-e^{-\kappa(s-t)}\right)^2}{\kappa^2} + \frac{1}{2}\sigma_V^4\theta_t \frac{\left(1-e^{-\kappa(s-t)}\right)^3}{\kappa^2} \quad (61) \\ &\quad + e^{-3\kappa s}\int_t^s 3\sigma_V^2 e^{\kappa u}\int_t^u E_t^Q(\lambda_v y^2)e^{2\kappa v}dvdu \\ &\quad + 3e^{-3\kappa s}\int_t^s e^{3\kappa u}E_t^Q(\lambda_u X_u y^2)du + e^{-3\kappa s}\int_t^s e^{3\kappa u}E_t^Q(\lambda_u y^3)du, \quad (62) \end{aligned}$$

which is the third moment provided in the Proposition 1.

B Proof of Proposition 3

Expanding $f(\theta_T, V_T)$ with the Taylor expansion near the point of (θ_t, V_T), we have

$$f(\theta_T, V_T) = f(\theta_t, V_T) + (\theta_T - \theta_t)f_\theta(\theta_t, V_T) + (\theta_T - \theta_t)^2\frac{1}{2}f_{\theta\theta}(\theta_t, V_T) + O(\sigma_\theta^4), \quad (63)$$

where $O(\sigma_\theta^4)$ stands for the terms with the order of σ_θ^4. Then, using $E_t^Q(\theta_T) = \theta_t$ and $E_t^Q\left[(\theta_T-\theta_t)^2\right] = \sigma_\theta^2(T-t)$, the time t expectation of $f(\theta_T, V_T)$ is

$$E_t^Q\left[f(\theta_T,V_T)\right] = E_t^Q\left[f(\theta_t,V_T)\right] + \sigma_\theta^2(T-t)\frac{1}{2}E_t^Q\left[f_{\theta\theta}(\theta_t,V_T)\right] + O(\sigma_\theta^4) \tag{64}$$

Further expanding $f(\theta_t, V_T)$ and $f_{\theta\theta}(\theta_t, V_T)$ near the point of $E_t^Q(V_T)$, we obtain

$$\begin{aligned} f(\theta_t,V_T) &= f(\theta_t,E_t^Q(V_T)) + (V_T - E_t^Q(V_T))f_V(\theta_t,E_t^Q(V_T)) \\ &\quad + (V_T - E_t^Q(V_T))^2\frac{1}{2}f_{VV}(\theta_t,E_t^Q(V_T)) + (V_T - E_t^Q(V_T))^3\frac{1}{6}f_{VVV}(\theta_t,E_t^Q(V_T)) \\ &\quad + O\left((V_T - E_t^Q(V_T))^4\right), \end{aligned} \tag{65}$$

and

$$\begin{aligned} f_{\theta\theta}(\theta_t,V_T) &= f_{\theta\theta}(\theta_t,E_t^Q(V_T)) + (V_T - E_t^Q(V_T))f_{\theta\theta V}(\theta_t,E_t^Q(V_T)) \\ &\quad + O\left((V_T - E_t^Q(V_T))^2\right). \end{aligned} \tag{66}$$

Taking conditional expectation on both sides of equations (65) and (66),

$$\begin{aligned} E_t^Q\left[f(\theta_t,V_T)\right] &= f(\theta_t,E_t^Q(V_T)) + \frac{1}{2}f_{VV}(\theta_t,E_t^Q(V_T))M_2(T) + \frac{1}{6}f_{VVV}(\theta_t,E_t^Q(V_T))M_3(T), \\ &\quad + O\left((V_T - E_t^Q(V_T))^4\right), \end{aligned} \tag{67}$$

and

$$E_t^Q\left[f_{\theta\theta}(\theta_t,V_T)\right] = f_{\theta\theta}(\theta_t,E_t^Q(V_T)) + O\left((V_T - E_t^Q(V_T))^2\right). \tag{68}$$

Therefore,

$$\begin{aligned} E_t^Q\left[f(\theta_T,V_T)\right] &= f(\theta_t,E_t^Q(V_T)) + \frac{1}{2}f_{\theta\theta}(\theta_t,E_t^Q(V_T))\sigma_\theta^2(T-t) \\ &\quad + \frac{1}{2}f_{VV}(\theta_t,E_t^Q(V_T))M_2(T) + \frac{1}{6}f_{VVV}(\theta_t,E_t^Q(V_T))M_3(T), \end{aligned} \tag{69}$$

where terms with order $O(\sigma_V^6)$ have been ignored, and we assume $O(\sigma_\theta) \sim O(\sigma_V^2)$.

C Proof of Proposition 4

Using the expression for $VIX_{t,\tau}$ in Proposition 2, we have

$$\begin{aligned} Y_{t,T} &\equiv \ln \frac{VIX_{T,\tau}}{VIX_{t,\tau}}, \\ &= \ln \frac{\sqrt{(1-\alpha)\theta_T + \alpha V_T}}{\sqrt{(1-\alpha)\theta_t + \alpha V_t}}, \\ &= \frac{1}{2}\{\ln[(1-\alpha)\theta_T + \alpha V_T] - \ln[(1-\alpha)\theta_t + \alpha V_t]\}. \end{aligned} \tag{70}$$

For simplicity, we let

$$U_T = \ln\left[(1-\alpha)\theta_T + \alpha V_T\right]. \tag{71}$$

Using the result in Proposition 3, we have

$$\begin{aligned} E_t^Q(U_T) &= \ln\left[(1-\alpha)\theta_t + \alpha E_t^Q(V_T)\right] \\ &\quad -\frac{1}{2}\left[(1-\alpha)\theta_t + \alpha E_t^Q(V_T)\right]^{-2}\alpha^2 M_2 + \frac{1}{3}\left[(1-\alpha)\theta_t + \alpha E_t^Q(V_T)\right]^{-3}\alpha^3 M_3 \\ &\quad -\frac{1}{2}(1-\alpha)^2\sigma_\theta^2(T-t)\left[(1-\alpha)\theta_t + \alpha E_t^Q(V_T)\right]^{-2}. \end{aligned} \tag{72}$$

$$\begin{aligned} E_t^Q\left(U_T^2\right) &= \ln^2 b + \left(\frac{1-\alpha}{b}\right)^2(1-\ln b)\sigma_\theta^2(T-t) \\ &\quad + \left(\frac{\alpha}{b}\right)^2(1-\ln b)M_2 + \frac{1}{3}\left(\frac{\alpha}{b}\right)^3(-3+2\ln b)M_3 + O\left(\sigma_V^6\right). \end{aligned} \tag{73}$$

$$\begin{aligned} E_t^Q(U_T^3) &= \ln^3 b + \frac{3}{2}\left(\frac{1-\alpha}{b}\right)^2(2\ln b - \ln^2 b)\sigma_\theta^2(T-t) \\ &\quad + \frac{3}{2}\left(\frac{\alpha}{b}\right)^2(2\ln b - \ln^2 b)M_2 + \left(\frac{\alpha}{b}\right)^3(1 - 3\ln b + \ln^2 b)M_3 + O\left(\sigma_V^6\right). \end{aligned} \tag{74}$$

Therefore, the first moment of $Y_{t,T}$ is given by

$$\begin{aligned} \bar{M}_1 &\equiv E_t^Q(Y_{t,T}), \\ &= \frac{1}{2}\{E_t^Q(U_T) - \ln\left[(1-\alpha)\theta_t + \alpha V_t\right]\}, \end{aligned} \tag{75}$$

where $E_t^Q(U_T)$ is given by equation (72).

The second moment of $Y_{t,T}$ can be obtained as

$$\begin{aligned}
\bar{M}_2 &\equiv E_t^Q\left[\left(Y_{t,T} - E_t^Q(Y_{t,T})\right)^2\right] \\
&= \frac{1}{4}E_t^Q\left[\left(U_T - E_t^Q(U_T)\right)^2\right] \\
&= \frac{1}{4}\left[\left(\frac{1-\alpha}{b}\right)^2 \sigma_\theta^2(T-t) + \left(\frac{\alpha}{b}\right)^2 M_2 - \left(\frac{\alpha}{b}\right)^3 M_3\right]. \qquad (76)
\end{aligned}$$

The third moment is given by

$$\begin{aligned}
\bar{M}_3 &\equiv E_t^Q\left[\left(Y_{t,T} - E_t^Q(Y_{t,T})\right)^3\right] \\
&= \frac{1}{8}E_t^Q\left[\left(U_T - E_t^Q(U_T)\right)^3\right] \\
&= \frac{1}{8}\left[E_t^Q\left(U_T^3\right) - 3E_t^Q(U_T)\,E_t^Q\left(U_T^2\right) + 2\left(E_t^Q(U_T)\right)^3\right]. \qquad (77)
\end{aligned}$$

where $E_t^Q(U_T^3)$, $E_t^Q(U_T^2)$ and $E_t^Q(U_T)$ are given by equations (74), (73) and (72), respectively.

References

[1] Adrian, Tobias, and Joshua Rosenberg, 2008, Stock returns and volatility: Pricing the short-run and long-run components of market risk, *Journal of Finance* 63, 2997-3030.

[2] Buehler, Hans, 2006, Consistent variance curve models, *Finance and Stochastics* 10, 178-203.

[3] Carr, Peter, and Roger Lee, 2009, Volatility derivatives, *Annual Review of Financial Economics* 1, 1-21.

[4] Chernov, Mikhail, and Eric Ghysels, 2000, Towards a unified approach to the joint estimation of objective and risk neutral measures for the purpose of options valuation, *Journal of Financial Economics* 56, 407-458.

[5] Christoffersen, Peter F., Steven Heston, and Kris Jacobs, 2009, The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well, *Management Science* 55, 1914-1932.

[6] Christoffersen, Peter F., Kris Jacobs, Chayawat Ornthanalai, and Yintian Wang, 2008, Option valuation with long-run and short-run volatility components, *Journal of Financial Economics* 90, 272-297.

[7] Cont, Rama, Thomas Kokholm, 2009, A consistent pricing model for index options and volatility derivatives, Working paper, Columbia University Center for Financial Engineering.

[8] Duffie, Darrel, Jun Pan, and Kenneth J. Singleton, 2000, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* 68, 1343-1376.

[9] Egloff, Daniel, Markus Leippold, and Liuren Wu, 2010, The term structure of variance swap rates and optimal variance swap investments, *Journal of Financial and Quantitative Analysis*, (forthcoming).

[10] Eraker, B., 2004, Do stock prices and volatility jump? Reconciling evidence from spot and option prices, *Journal of Finance* 58, 1269-1300.

[11] Gatheral, Jim, 2008, Consistent modeling of SPX and VIX options, in Bachelier Congress, London.

[12] Heston, Steven, 1993, Closed-form solution of options with stochastic volatility with application to bond and currency options, *Review of Financial Studies* 63, 327-804.

[13] Jones, C. S., 2003, The dynamics of stochastic volatility: Evidence from underlying and options markets, *Journal of Econometrics* 116, 181-224.

[14] Lin, Yueh-Neng, 2007, Pricing VIX futures: Evidence from integrated physical and risk-neutral probability measures, *Journal of Futures Markets* 27, 1175-1217.

[15] Lin, Yueh-Neng, and Chien-Hung Chang, 2009, VIX option pricing, *Journal of Futures Markets* 29, 523-543.

[16] Lin, Yueh-Neng, and Chien-Hung Chang, 2010, Consistent modeling of S&P 500 and VIX derivatives, *Journal of Economic Dynamics and Control*, (forthcoming).

[17] Lu, Zhongjin, and Yingzi Zhu, 2010, Volatility components: The term structure dynamics of VIX futures, *Journal of Futures Markets* 30, 230-256.

[18] Luo, Xingguo, and Jin E. Zhang, 2009, The term structure of VIX, Working paper, The University of Hong Kong.

[19] Mencia, Javier, and Enrique Sentana, 2009, Valuation of VIX derivatives, CEMFI Working paper 0913.

[20] Pan, Jun, 2002, The jump-risk premia implicit in options: Evidence from an integrated time-series study, *Journal of Financial Economics* 62, 3-50.

[21] Sepp, Artur, 2008a, VIX option pricing in jump-diffusion model, *Risk*, April 2008, 84-89.

[22] Sepp, Artur, 2008b, Pricing options on realized variance in Heston model with jumps in returns and volatility, *Journal of Computational Finance* 11, 33-70.

[23] Wang, Zhiguang, and Robert T. Daigler, 2010, The performance of VIX option pricing models: Empirical evidence beyond simulation, *Journal of Futures Markets*, (forthcoming).

[24] Zhang, Jin E., Jinghong Shu, and Menachem Brenner, 2010, The new market for volatility trading, *Journal of Futures Markets* 30, 809-833.

[25] Zhang, Jin E., and Yi Xiang, 2008, The implied volatility smirk, *Quantitative Finance* 8, 263-284.

[26] Zhang, Jin E., and Yingzi Zhu, 2006, VIX futures, *Journal of Futures Markets* 26, 521-531.

[27] Zhu, Yingzi, and Jin E. Zhang, 2007, Variance term structure and VIX futures pricing, *International Journal of Theoretical and Applied Finance* 10, 111-127.

Table 1: **Descriptive Statistics for Daily VIX Option Implied Volatility**

This table provides descriptive statistics for the daily VIX option implied at the money volatilities with maturities 2, 3, 4, 5, and 6 months. Reported are the mean, standard deviation, skewness, kurtosis, minimum and maximum. The sample period is from March 22, 2006 to June 8, 2007.

Maturity	Mean	Std.dev.	Skewness	Kurtosis	Minimum	Maximum
2-m	0.711	0.099	-0.111	3.327	0.437	1.015
3-m	0.654	0.078	-0.214	2.963	0.454	0.876
4-m	0.611	0.062	-0.540	2.821	0.460	0.743
5-m	0.577	0.052	-0.516	3.194	0.435	0.697
6-m	0.554	0.044	-0.443	3.170	0.431	0.654

Table 2: **Descriptive Statistics for Daily VIX Term Structure Data**

This table provides descriptive statistics for the daily VIX term structure data with maturities 2, 3, 4, 5, and 6 months. Reported are the mean, standard deviation, skewness, kurtosis, minimum and maximum. The sample period is from March 22, 2006 to June 8, 2007.

Maturity	Mean	Std.dev.	Skewness	Kurtosis	Minimum	Maximum
2-m	0.133	0.018	1.04	4.14	0.104	0.211
3-m	0.138	0.016	0.97	4.16	0.113	0.208
4-m	0.140	0.015	0.89	3.93	0.118	0.202
5-m	0.142	0.013	0.86	3.83	0.122	0.197
6-m	0.144	0.013	0.84	3.78	0.124	0.194

Figure 1: **Time Series of VIX Option Implied at the Money Volatilities**

We show time series of the daily VIX option implied at the money volatilities with maturities of 2, 4, and 6 months, from March 22, 2006 to June 8, 2007.

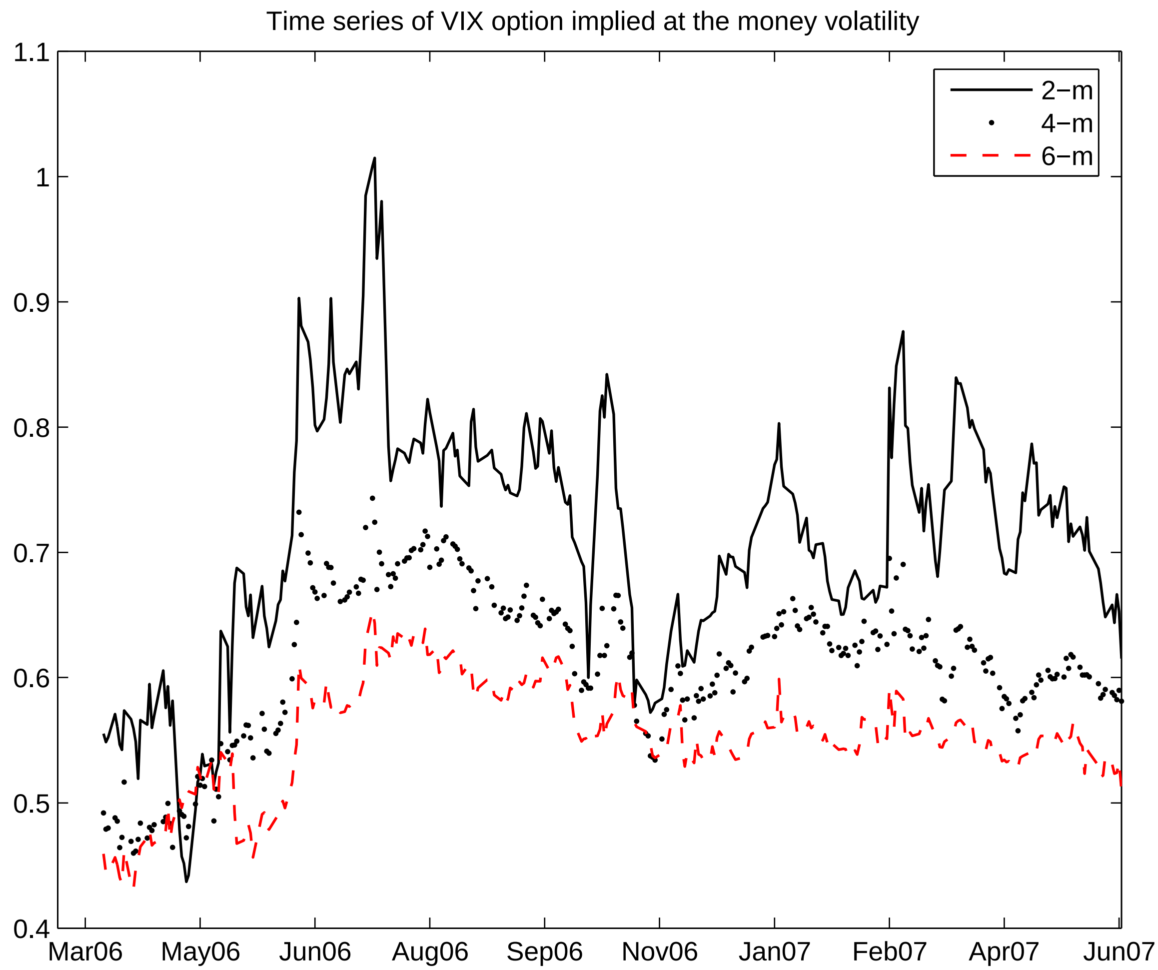

Figure 2: **Time Series of VIX Term Structure**

We show time series of the daily VIX term structure data with maturities of 2, 4, and 6 months, from March 22, 2006 to June 8, 2007.

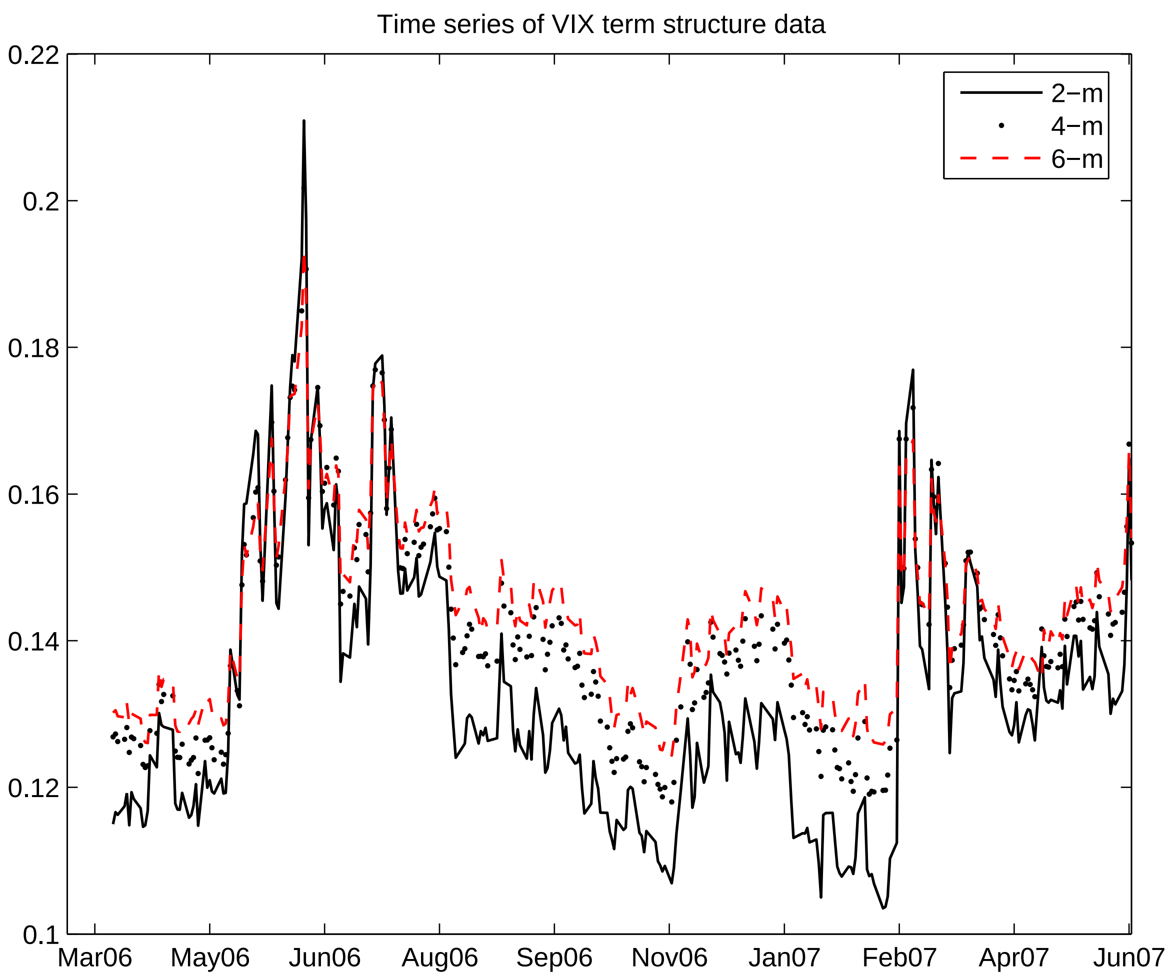

Figure 3: **Time Series of the Estimated Instantaneous Variance and its Long Term Mean Level**

We show time series of the daily estimated instantaneous variance (dotted red lines), V_t, and its long term mean level (black lines), θ_t, from March 22, 2006 to June 8, 2007.