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<td><strong>Issued Date</strong></td>
<td>2011</td>
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<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/10722/135896">http://hdl.handle.net/10722/135896</a></td>
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Robust Stabilization of a Class of Nonlinear Systems with Uncertain Parameters Based on CLFs

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Abstract: This paper is considered with the robust stabilization problem of a class of nonlinear systems with bounded uncertain time-invariant parameters. A robust control Lyapunov function (RCLF) is introduced for the considered system. Based on the RCLF, a globally asymptotically stabilizing controller is then designed. The proposed controller is robust under the variant of system parameters. As the applications of the proposed scheme, the stabilization of uncertain feedback linearizable systems and the unified chaotic system are investigated, respectively. A numerical example on the unified chaotic system is also provided to illustrate the effectiveness of the presented method.

Key Words: Stabilization, Nonlinear systems, Robust control Lyapunov functions, Parameter uncertainty, Feedback linearizable systems, Chaotic systems

1 Introduction

Recent years have witnessed an increasing interest in the design of nonlinear control systems with uncertain parameters [2], [7]-[11], [17], [18], [21]-[24], partly due to the desirable requirements on guarantees of stability and performance against uncertainty on the physical parameters of the system. Numerous of methods, such as the robust stabilization [17], [22], adaptive control [11], [21], backstepping design [18], observer-based control [26], and passive method [5], [10], have been developed to solve this problem. In general, adaptive method and back-stepping are two popular design tools used in existing literature [2], [7], [11]. For example, in [26] and [24], the observer-based adaptive design and the backstepping approach are respectively used to deal with the synchronization problem of chaotic systems with unknown parameters.

Instead of using the adaptive control or back-stepping, in this paper an approach based on control Lyapunov function (CLF) is proposed to design the stabilizing state feedback. It is known that the concept of CLF introduced by Artstein [1] and Sontag [15] is a powerful tool in the design theory of nonlinear feedback stabilization. Many useful results for determined nonlinear systems have been obtained by using CLFs [2]. Recently, several authors tried to extend these results to uncertain nonlinear systems [9], [11], [13]. A CLF based designs for nonlinear systems with input uncertainties is proposed in [9]. Krstic and Kokotovic [12] introduced the adaptive CLF (ACLF) for the problem of stabilization of affine nonlinear systems with uncertain parameters. Cai and Han studied the robust stabilization problem of nonlinear systems with structure uncertainty [3]-[5]. Wu [23] and Zhang et al. [25] used the CLF to investigate the simultaneous stabilization for a collection of single-input nonlinear systems. Based on the CLF, robust stabilization of a class of single-input polytopic nonlinear systems was investigated by Wu in [22].

In this paper, we will further study the robust stabilization of a class of nonlinear systems with bounded uncertain time-invariant parameters by using CLFs. A robust control Lyapunov function (RCLF) is introduced for such systems. Based on the RCLF, we then design a globally asymptotically stabilizing control law, which is robust under the variant of the bounded parameters.

The paper is organized as follows. In Section 2, a concept of robust control Lyapunov function (RCLF) is firstly introduced for a class of nonlinear systems with bounded uncertain parameters. In Section 3, a Sontag’s formula-like control law is proposed. In Section 4, as the applications of the presented method, the stabilization of uncertain feedback linearizable systems and the unified chaotic system are studied, respectively. An illustrative numerical example of controlling unified chaotic system is also given in Section 4. Conclusions are drawn in Section 5.

2 Preliminaries

Consider a class of nonlinear systems described by

\[ \dot{x} = f(x, \eta) + \sum_{i=1}^{m} g_i(x)u_i, \]

where \( x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n \) is state of the system and \( u = [u_1, u_2, \ldots, u_m]^T \in \mathbb{R}^m \) is control input; \( \eta \) is a time-invariant parameter vector of the system. We assume that \( \eta \) varies in a compact set \( \Omega \) of \( \mathbb{R}^p \), where \( p \) is a determined integer. The functions \( f(x, \eta) \) and \( g_i(x) (i = 1, 2, \ldots, m) \) are assumed sufficient smooth on their arguments and \( f(0, \eta) = 0 \) for all \( \eta \in \Omega \).
Our goal is to design a state feedback law \( u = p(x) \) such that the corresponding closed-loop system is globally asymptotically stable. To begin with, we introduce the concept of robust control Lyapunov function (RCLF), which follows the standard definition of CLFs due to Sontag [16].

**Definition 1.** A smooth, positive definite and radially unbounded function \( V(x) \) is a RCLF for system (1) if for any \( x \in \mathbb{R}^n \) and \( \eta \in \Omega \),

\[
B(x) = 0, \quad x \neq 0 \quad \Rightarrow \quad a(x, \eta) < 0,
\]

(2)

where

\[
a(x, \eta) = \frac{\partial V(x)}{\partial x} f(x, \eta), \quad B(x) = \frac{\partial V(x)}{\partial x} G(x),
\]

(3)

and

\[
G(x) = [g_1(x), g_2(x), \ldots, g_m(x)]^T.
\]

(4)

**Definition 2.** A RCLF of system (1) satisfies the small control property (SCP) if for each \( \epsilon > 0 \), there is a \( \delta > 0 \) such that, if \( x \neq 0 \) satisfies \( \|x\| < \delta \), then there is some \( u \) with \( \|u\| < \epsilon \) such that \( a(x, \eta) + B(x)u < 0 \), where \( a(x, \eta) \) and \( B(x) \) are defined by (3)-(5).

**Remark 1.** By Artstein-Sontag theorem [8], system (1) is globally asymptotically stabilizable by an almost smooth state feedback if and only if it has a RCLF satisfying the SCP. If there is a RCLF \( V(x) \) for system (1), we can construct the state feedback control law by using the so-called Sontag's universal formula [16].

**Remark 2.** In general, finding a RCLF for system (1) is not an easy task or even possible. Fortunately, in many cases of interesting we can construct the RCLF indeed as we will see below. Moreover, the following Lemma 1 is very helpful for the construction of RCLFs. We omit the proof here since it can be deduced directly by using Definition 1.

**Lemma 1.** Let \( K(x) \in \mathbb{R}^m \) be a continuous function. If the feedback takes the form of \( u = K(x) + v \), where \( v \) stands for the new control input, then \( V(x) \) is a RCLF for the following system

\[
\dot{x} = f(x, \eta) + G(x)K(x) + g(x)v
\]

(6)

and if only if it is a RCLF for system (1).

**Lemma 2.** If \( g(x, d) \) is a continuous function on \( \mathbb{R}^n \times \Omega \), where \( \Omega \) is a compact set of \( \mathbb{R}^n \), then

\[
\tilde{g}(x) \triangleq \max_{d \in \Omega} g(x, d)
\]

is a continuous function.

**Proof.** Let \( x_0 \in \mathbb{R}^n \) and \( \{x_m\} \) be a series on \( \mathbb{R}^n \) such that \( x_m \to x_0 (m \to \infty) \). Then for every \( x_m \), there exists a \( d_m \in \Omega \) such that

\[
\tilde{g}(x_m) = g(x_m, d_m) = \max_{d \in \Omega} g(x_m, d).
\]

(7)

We then obtain a series of \( \{d_m, m = 1, 2, \ldots\} \subset \Omega \). Since \( \Omega \) is compact, \( \{d_m\} \) contains a convergent subseries \( \{d_{m_k}\} \). Let \( d^* \) be the limit of \( \{d_{m_k}\} \).

On the other hand, by the definition of \( \tilde{g}(x_0) \), there exists a \( d_0 \) such that

\[
\tilde{g}(x_0) = \max_{d \in \Omega} g(x_0, d) = g(x_0, d_0).
\]

Now let us consider a subseries \( \{x_{m_k}\} \) of \( \{x_m\} \). Since \( \{x_{m_k}\} \) is convergent, we have \( x_{m_k} \to x_0 (k \to \infty) \). By the continuity of \( g(x, d) \), we know that for every \( \varepsilon > 0 \), there exists an integer \( K \), such that when \( k > K, g(x_{m_k}, d_0) + \varepsilon > g(x_0, d_0) \). Thus, we have

\[
g(x_{m_k}, d_{m_k}) + \varepsilon \geq g(x_{m_k}, d_0) + \varepsilon \\
\geq g(x_0, d_0) \geq g(x_0, d_{m_k}).
\]

(8)

Let \( k \to \infty \). Then from (8) we get

\[
g(x_0, d^*) + \varepsilon \geq g(x_0, d_0) \geq g(x_0, d^*)
\]

(9)

Note that \( \varepsilon > 0 \) is selected arbitrarily. Therefore, (9) implies

\[
\tilde{g}(x_0) = g(x_0, d_0) = g(x_0, d^*)
\]

By the continuity of \( g(x, d) \), we have

\[
\tilde{g}(x_0) = g(x_0, d^*) = \lim_{k \to \infty} g(x_{m_k}, d_{m_k}) = \lim_{k \to \infty} \tilde{g}(x_{m_k})
\]

In order to complete the proof, we now verify the conclusion \( \lim_{m \to \infty} \tilde{g}(x_m) = \tilde{g}(x_0) \) by contradiction.

If \( \{x_{m_k}\}, m = 1, 2, \ldots \) is not convergent, then there exists a subseries \( \{x_{m_{l_k}}\}, l = 1, 2, \ldots \) and \( \lim_{l \to \infty} x_{m_{l_k}} = x_0 \) such that \( \lim_{l \to \infty} \tilde{g}(x_{m_{l_k}}) = g(x_0) \), where \( g(x_0) \) may be a real number or infinity, but \( g(x_0) \neq g(x_0) \). In order to simplify the notation, we denote \( x_{m_{l_k}} \) by \( x_l \).

Now let us repeat the previous procedure. For every \( x_l \), there exists a \( d_l \in \Omega \) such that

\[
\tilde{g}(x_l) = \tilde{g}(x_l, d_l) = \max_{d \in \Omega} g(x_l, d).
\]

Similarly, we can obtain a series \( \{d_l, l = 1, 2, \ldots\} \subset \Omega \) and then a convergent subseries \( \{d_{l_k}\} \) are obtained. By the same processing, it is obtained that \( \lim_{l \to \infty} \tilde{g}(x_{l_k}) = \tilde{g}(x_0) \). Because \( \tilde{g}(x_l) = g(x_l, d_l) \) is convergent, \( \lim_{l \to \infty} \tilde{g}(x_{l_k}) = \tilde{g}(x_0) \). A contradiction then happens. Thus, we conclude that \( \lim_{m \to \infty} \tilde{g}(x_m) = \tilde{g}(x_0) \), i.e., \( g(x_0) \) is a continuous function. \( \square \)

### 3 Design of Robust Stabilizing Controller

This section presents a robust stabilizing controller for system (1) by using the RCLF. We sum it up as the following theorem.

**Theorem 1.** Assume that \( V(x) \) satisfying SCP is a RCLF for system (1). Then there exists a state feedback which can globally asymptotically stabilize system (1) for every \( \eta \in \Omega \).

**Proof.** Let \( V(x) \) satisfying SCP be a RCLF of system (1). Let \( a(x, \eta) \) and \( B(x) \) be defined as in (3)-(5). Then \( V(x) = a(x, \eta) + B(x)u \). The feedback under consideration is

\[
u = p(x) = \begin{cases} 
-\phi(x) & \text{if } B(x) \neq 0, \\
0 & \text{if } B(x) = 0.
\end{cases}
\]

(10)

where

\[
\phi(x) = \frac{\bar{a}(x) + \sqrt{\bar{a}(x)^2 + \|B^T(x)\|^2}}{B(x)B^T(x)} B^T(x)
\]

and

\[
\bar{a}(x) = \max_{\eta \in \Omega} \{a(x, \eta)\}.
\]
From Lemma 2 and the small control property of \( V(x) \), we know that \( a(x) \) and the control law \( u \) are continuous on their augments. If \( B(x) \neq 0 \) and \( x \neq 0 \), then
\[
\dot{V}(x) = a(x, \eta) + B(x)p(x) = a(x, \eta) - \dot{a}(x) - \sqrt{a(x)} + \|B^T(x)\| \geq -\sqrt{a(x)} + \|B^T(x)\| < 0.
\]

When \( B(x) = 0 \), by the definition of RCLF, \( \dot{V}(x) = a(x, \eta) < 0 \). Therefore, if the state feedback takes the form of (10), we always have \( \dot{V}(x) < 0 \) for all \( x \neq 0 \), which implies that the corresponding close-loop system is globally asymptotically stable. This completes the proof. \( \square \)

**Remark 3.** By Theorem 1, we know that the design will become easy if we can construct a RCLF for system (1). Generally, constructing the RCLF for an ordinary nonlinear system became easy if we can construct a RCLF for system (1). For instance, the monograph of Isidori [8] for details. In this section, we consider a class of uncertain feedback linearizable systems and the unified chaotic sections, as the applications of Theorem 1, we will use the RCLF indeed as we will see below. In the next two sections, as the applications of Theorem 1, we will use the RCLF to deal with the robust stabilization of a class of uncertain feedback linearizable systems and the unified chaotic system, respectively.

### 4 Robust Stabilization of Feedback Linearizable Systems

It is known that feedback linearizable systems are frequently used in the study of nonlinear control systems. See, for instance, the monograph of Isidori [8] for details. In this section, we consider a class of uncertain feedback linearizable systems described by
\[
\dot{x} = Ax + B[\varphi(x, \eta) + \rho(x)u],
\]
where \( \eta \) is the system parameters varied in a compact set \( \Omega \), \( \rho(x) \neq 0 \) for all \( x \in \mathbb{R}^n \). \( (A, B) \) are assumed holds the control Brunovsky canonical form given by
\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix} 0 \\
0 \\
\vdots \\
1 \\
\end{bmatrix}.
\]

**Lemma 3.** System (11) holds a quadratic RCLF when \( \eta \) varies in a compact set \( \Omega \).

**Proof:** From Lemma 1, we have known that RCLF is invariant under state feedback and input transformation. By the feedback and input transformation of
\[
u = \rho^{-1}(x)[-\varphi(x, \eta) + v],
\]
the system (11) is transformed into a Brunovsky canonical form
\[
\dot{x} = Ax + Bu.
\]
If \( V(x) \) is a RCLF of (12), then it is also a RCLF of (11). It is not difficult to present a RCLF for (12). In fact, since \( (A, B) \) is controllable, there exists a matrix \( K \) such that \( A + BK \) is a Hurwitz matrix. Consequently, the following Lyapunov equation
\[
(A + BK)^TP + P(A + BK) = -Q,
\]
has a unique positive definite solution \( P \) for any positive definite matrix \( Q \). Hence, for any \( x \in \mathbb{R}^n \), we have
\[
x^T(A^TP + PA)x + x^TPBKx + x^TK^TPx = -x^TQx.
\]
Let \( V(x) = x^TPx \). Then, in view of (13), we have
\[
\frac{\partial V(x)}{\partial x}B = 2x^TPB = 0, \quad x \neq 0 \Rightarrow \frac{\partial V(x)}{\partial x}Ax = x^T(A^TP + PA)x = -x^TQx < 0,
\]
which implies that \( V(x) \) is a RCLF for system (12). And then, according to Lemma 1, \( V(x) \) is a RCLF for system (11) as well. This completes the proof. \( \square \)

Note that the proof of Lemma 3 also presents a simple method of constructing RCLFs for the uncertain feedback linearizable system of the form (11). Based on Lemma 3 and Theorem 1, we can obtain the following conclusion.

**Theorem 2.** The uncertain feedback linearizable system of the form (11) can be globally asymptotically stabilized via a state feedback controller.

### 5 Controlling of Uncertain Chaotic Systems

In this section, we consider the problem of controlling of uncertain chaotic dynamical systems. This problem is one of the main topics in nonlinear dynamics. Many effective methods, such as inverse optimal control, adaptive control, and robust control, have been developed for this issue in the past decade [6], [19]. Here we study this issue by applying the technique developed in Section 3. For the sake of simplicity, we take the unified chaotic system as an illustrative example. Recently, Lü et al. [14] presented a so-called unified chaotic system as follows:

\[
\begin{align*}
t_1 &= (25\theta + 10)(x_2 - x_1) \\
t_2 &= (28 - 35\theta)x_1 - x_1x_3 + (29\theta - 1)x_2 \\
t_3 &= x_1x_2 - (8 + \theta)x_3/3
\end{align*}
\]

where \( x = (x_1, x_2, x_3)^T \) is the state, \( \theta \in [0, 1] \) is a parameter of the system. As introduced in [14], when \( \theta \in [0, 0.8] \), system (15) is called the generalized Lorenz chaotic system, and when \( \theta = 0.8 \), system (15) is called a Lü chaotic system (see, e.g., Fig. 3), and when \( \theta \in (0.8, 1] \), system (15) is called the generalized Chen chaotic system. For system (15), we impose a control on the second term such that it becomes the following controlled system of the compact form
\[
\dot{x} = f(x, \theta) + Bu,
\]
where
\[
f(x, \theta) = \begin{pmatrix}
(25\theta + 10)(x_2 - x_1) \\
(28 - 35\theta)x_1 - x_1x_3 + (29\theta - 1)x_2 \\
x_1x_2 - (8 + \theta)x_3/3
\end{pmatrix}
\]
and
\[
B = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}.
\]
Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$. Then the time derivative of $V(x)$ along the trajectories of system (16) is

$$\dot{V} \big|_{(16)} = (x_1 x_2 x_3) f(x, \theta) + x_2 u = a(x, \theta) + x_2 u,$$

where $a(x, \theta) = (x_1 x_2 x_3) f(x, \theta)$. Notice that for every $\theta \in [0, 1]$, when $x_2 = 0$ and $x \neq 0$,

$$a(x, \theta) = -(25 \theta + 10)x_1^2 - \frac{1}{3}(8 + \theta)x_3^2 < 0.$$

This implies that $V(x)$ is a RCLF for system (16). Then based on Theorem 1, the following state feedback

$$u = k(x) = \begin{cases} \frac{-\bar{a}(x) + \sqrt{\bar{a}(x)^2 + x_2^2}}{x_2} & \text{if } x_2 \neq 0, \\ 0 & \text{if } x_2 = 0. \end{cases}$$

(17)

where $\bar{a}(x) = \max_{\theta \in [0, 1]} \{a(x, \theta)\}$, can globally asymptotically stabilize system (16) when the parameter $\theta$ varies in $[0, 1]$. Figs. 1 and 2 display the state trajectories and control law of system (16) with initial state $x_0 = (1 - 1/3)^2$ and $\theta = 0.75$, respectively. The simulation result shows that the effectiveness of the proposed method.

**Remark 4.** It is worth to point out that the design method provided in Section 3 can be readily extended to solve the chaos synchronization problem. A similar method used the CLF approach to synchronization of unified chaotic system with uncertain parameters can be found in a recent paper of Wang et al. [20].

**6 Conclusions**

We have presented an effective method to study the robust stabilization of a class of nonlinear systems with bounded uncertain parameters. Firstly, based on control Lyapunov function, a globally asymptotically stabilizing state feedback controller is designed. Subsequently, as the applications of the proposed scheme, the stabilization of uncertain feedback linearizable systems and unified chaotic systems are studied, respectively. Finally, a numerical example is provided to illustrate the effectiveness of the presented method.

**References**


