Distribution-free Travel Time Reliability Assessment with Probability Inequalities

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Abstract

An assumption that pervades the current transportation system reliability assessment literature is that probability distributions of the sources of uncertainty are known explicitly. However, this distribution may be unavailable (inaccurate) in reality as we may have no (insufficient) data to calibrate the distribution. In this paper we relax this assumption and present a new method to assess travel time reliability that is distribution-free in the sense that the methodology only requires that the first N moments (where N is a user-specified positive integer) of the travel time to be known and that the travel times reside in a set of bounded and known intervals. Because of our modeling approach, all sources of uncertainty are automatically accounted for, as long as they are statistically independent. Instead of deriving exact probabilities on travel times exceeding certain thresholds via computationally intensive methods, we develop semi-analytical probability inequalities to quickly (i.e. within a fraction of a second) obtain upper bounds on the desired probability. Numerical experiments suggest that the inclusion of higher order moments can potentially significantly improve the bounds. The case study also demonstrates that the derived bounds are nontrivial for a large range of travel time values.

Keywords**:** uncertainty, travel time reliability, probability inequality, bounds, independence, moments.

1. Introduction

Real-life transportation systems are characterized by a number of uncertainties, both from the supply side (e.g. road capacity variations) as well as from the demand side (i.e. Origin-Destination (OD) travel demand variations). In an attempt to characterize the performance of a transportation system, several reliability measures have been proposed in the literature. One of the measures is given by connectivity or terminal reliability (Wakabayashi and Iida, 1992; Bell and Iida, 1997; Asakura et al., 2001) that concerns the probability that specific OD pairs in a network remain connected when links are subject to complete failures. As such, this measure is most appropriate for the modeling of extreme events such as earthquakes. Another measure of reliability was proposed in Chen et al. (1999, 2002) under the name capacity reliability, which is defined as the probability that the transportation system can accommodate a given demand level at an acceptable level of service. More recently, Heydecker et al. (2007) proposed the use of the travel demand satisfaction reliability to measure road network performance. In this paper we adopt another widely popular measure of reliability: travel time reliability.

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Travel time reliability relates to the probability that travel times remain below acceptable levels. The earliest studies in this area employed extensive computer simulations in which a traffic assignment problem was solved for each scenario to determine the reliability of travel times (e.g. Asakura and Kashiwadani, 1991). Later studies employed various techniques and assumptions in an attempt to reduce the computational burden. For example, Du and Nicholson (1997) and Bell et al. (1999) employed sensitivity analysis to reduce the computational time. Sumalee and Watling (2003, 2008) proposed to compute bounds instead of exact probabilities. In Clark and Watling (2005) it is assumed that travelers simply do not have enough time to react to the unfolding scenario because of the unpredictability of events (which eliminated the need to solve a traffic assignment problem for each scenario). Following Lo and Tung (2003), who suggested that people learn about the uncertainties based on which they settle in a fixed, long-term equilibrium pattern, Ng and Waller (2010) presented a computationally efficient methodology based on the Fast Fourier Transform (FFT) to construct the entire probability distribution function of the system travel time under independent capacity variations. As such, their work can be seen as a complement of Clark and Watling (2005) that derived the probability distribution function of the system travel time under demand uncertainty. A fundamentally different approach that utilizes game theory to assess travel time reliability has been taken by, for instance, Bell (2000) and Szeto et al. (2009). Network equilibrium models that account for uncertainty can also be found in the literature, for example, Szeto et al. (2006), Lam et al. (2008), Siu and Lo (2008), Chen and Zhou (2010), Sumalee et al. (2010) and Zhang et al. (2010).

An assumption that pervades the current transportation system reliability assessment literature is that probability distributions are known explicitly. For instance, Chen et al. (2002) assumed that road capacities follow some pre-defined probability distributions with known correlation structures (for a comprehensive review of this correlation-based approach, see Ng et al., 2010a); in Clark and Watling (2005) the joint distribution of the link flows is given by the multivariate normal distribution. While the methodology developed in Ng and Waller (2010) is not limited to any particular distribution, in order to use the methodology, probability distributions for the road capacities need to be specified. Such distributions might not be available in real-life, which has led us to relax this assumption in the current work. In particular, the proposed approach only requires the specification of the first N (where N is a user-specified positive integer) moments and a set of finite intervals in which the random quantities are hypothesized to reside. Exact probability distributions are not required to be known. However, this relaxation comes at a cost. Instead of obtaining exact probability statements (that are as good as the underlying hypothetical assumptions on the probability distributions), we obtain upper bounds on the tail probabilities that are valid under *all* conceivable probability distributions, provided that the sources of uncertainty are statistically independent. Consequently, the bounds can be interpreted as worst-case bounds, valid for all situations that are consistent with the input data (note that this same idea emerges in robust optimization as well, e.g., see Bertsimas and Sim, 2004). In the transportation reliability literature, the game theoretical approach proposed by Bell (2000) and co-workers also displays some similarity. However, unlike the game theoretical approach (where travelers are assumed to be extremely pessimistic about the state of the network and where there must be at least one link failure per OD pair), we do not impose any assumptions on travel behavior and the number of occurrences of link failures. Moreover, the output of our work is a bound on the worst-case travel time, whereas the output in game theoretical approaches is the *expected* travel time and its associated reliability under the above mentioned assumptions.

Computational efficiency is another desirable characteristic of our distribution-free approach. Because of the semi-analytical character of the bounds, the proposed approach is very computationally efficient: A single bound can be obtained within a fraction of a second in the current paper. However, to achieve this computational efficiency our approach only yields bounds as opposed to exact reliability statements (that are as accurate as the underlying model assumptions).

The remainder of this paper is organized as follows. Section 2 presents a description of the reliability problem considered in this paper. In Section 3 we collect some inequalities that are used repeatedly in this work. Bounds using the first order moments only are formally derived in Section 4. In Section 5 we develop a new class of bounds that can potentially include moments up to order N. Some convexity results with regard to the bounds are discussed in Section 6. A numerical case study using the Sioux Falls test network is conducted in Section 7. Conclusions and future research directions are given in Section 8.

2. Problem Definition

In the transportation reliability literature (e.g. Chen et al. 2002; Lo and Tung, 2003; Ng and Waller, 2009), the travel time on link *a* is often assumed to be given by the Bureau of Public Roads (BPR) volume-delay function (Bureau of Public Roads, 1964; Sheffi, 1985). Uncertainties in capacities (e.g. Lo and Tung, 2003; Ng and Waller, 2010) and link flows (e.g. Clark and Watling, 2005) then give rise to uncertainties in link travel times after a transformation by the BPR function. In this paper we depart from this modeling paradigm. We believe that in practice it is much easier to specify probability distributions/ moment information (in particular, the higher order moments) for travel times than for capacities, say. For example, we believe that it is much easier to determine the variance of travel time than the variance of a road's capacity. Hence, in this paper, we directly focus our attention on the travel time, rather than on the underlying random variable. Besides of the increased intuition, our modeling paradigm is also more general and tighter bounds can be obtained as well (cf. Ng, 2010). However, we want to note that by adopting this modeling paradigm, one is – for example – not able to explicitly consider uncertainty propagation through link performance functions (e.g. Ng and Waller, 2010), which might be a disadvantage of this modeling approach.

To be more precise about the contribution of the current paper, let us first introduce some mathematical notation. Let *A* denote the set of links in a transportation network (with some abuse of notation, we shall also refer to the cardinality of this set as *A*; from the context it should be clear what is meant). Furthermore, let T_a denote the total (random) travel time on a given link $a \in A$, i.e. the travel time experienced collectively by all travelers on the link. At a system level, one is then interested in the weighted total system travel time (TSTT):

$$T \equiv \sum_{a=1}^{A} w_a T_a \,,$$

where w_a are non-negative real numbers such that $\sum_{a=1}^{A} w_a = 1$ (the astute reader will notice that this restriction is not necessary for the further development of the paper, however, we believe that this condition gives a better intuition to the interpretation of the weights). In particular, when $w_a = 1/A$, T reduces to the familiar average total system travel time (Ng and Waller, 2010); using non-uniform values for w_a , one can capture the relative importance of the different roads in a network, e.g. one might put larger weights on major freeways than local streets. Alternatively, one can assume the perspective of a traveling individual, in which case T_a would denote the travel time experienced by a single traveler on a given link and the summation would be over a specific route rather than over the entire transportation network (as in Chen et al., 2002; Lo and Tung, 2003). Without loss of generality, we take the system-level perspective in this paper, following recent work in this field (such as Clark and Watling, 2005 and Ng and Waller, 2010). Moreover, consistent with the state-of-the-art in the (semi-)analytical transportation reliability literature (apart from a class of studies assuming multivariate normality, first introduced by Clark and Watling, 2005), we assume that T_a are independent random variables. While it is true that for certain situations stochastic dependencies are critical to account for (e.g. during floods and earthquakes), for other situations, the assumption of independence might be justifiable (e.g. traffic accidents, parking violations), as pointed out by Lo and Tung (2003). We do not impose any assumptions on the underlying source of uncertainty, provided that the uncertainties exhibit the property of statistical independence. (Note that demand/ link flow uncertainty can generally not be modeled within the current framework as they are inherently dependent due to flow conservation.). Of course, for concreteness, the reader is free to think of the random travel times as results of independent capacity variations due to, for example, minor traffic incidents (Lo and Tung, 2003; Ng and Waller, 2010). However, at the same time, the reader should bear in mind that the results in this paper are much more general. Finally, a remark on the estimation of the moments of T_a is in order. From the perspective of an individual traveler, it is relatively straightforward to estimate these moments (i.e., the moments of the travel times experienced by a single individual): simply use their sample counterparts based on, for example, loop detector data (e.g. as in Kwon et al, 2000) (cf. the method of moments in statistics, e.g. see Casella and Berger, 2002). From the system-level perspective (where T_a denotes the *total* link travel time), one needs to estimate the moments of the *product* of the above "individual travel time" and a link flow. Since link flows must be deterministic in the current paper (if we allow for random link flows, due to flow conservation, there would be dependencies in link travel times, violating the independence assumption in this paper), it suffices to estimate the moments of the "individual travel times" (this can be done as above) and the link flows (traffic detector data and/ or simulation might be used for this purpose), *separately*. Their *product* can then be used as an estimate of the moments of the *total* link travel time.

We are now ready to define our contribution mathematically. Let $R(x)$ denote the *unreliability* function of T at x:

$$R(x) \equiv \Pr\left(T > x\right).$$

In the probability literature, $R(x)$ is oftentimes referred to as the *reliability* function (Ross, 2002). However, unlike in the transportation reliability literature – where small values of $R(x)$ are desirable – in the probability literature, high values of $R(x)$ typically indicate "good" system performance (where T would for example correspond to the lifetime of a component of a machine). For this reason, we have chosen to call $R(x)$ the *unreliability* function. In this paper we derive nontrivial upper bounds on the unreliability function for a range of interesting x values, i.e. we find real numbers $r(x)$, $0 \le r(x) \le 1$ such that $R(x) \le r(x)$. Instead of entire probability distributions (Lo and Tung, 2003; Ng and Waller, 2010), we only assume that the first N moments of the travel time T_a are known and that each of T_a resides in a known and bounded interval.

3. Some Useful Inequalities

In this section we collect a number of inequalities that will be used repeatedly in Sections 4 and 5 to arrive at the main results of this work.

Lemma 1 Let $T_1, T_2, \ldots, T_A$ be non-negative random variables. Furthermore, let $w_1, w_2, \ldots, w_A$ denote non-negative real numbers and let $T = \sum_{a=1}^{A} w_a T_a$. Then for $\lambda > 0$ we have

$$\Pr\left(T > t\right) \le \inf_{\lambda > 0}\left[\exp(-\lambda t) E\left[\exp(\lambda \sum_{i=1}^{A} w_i T_i))\right]\right]$$

where $E[X]$ denotes the mathematical expectation of the random variable X.

Proof Since $T \ge 0$ with probability one and $\lambda > 0$, we have

$$\Pr\left(T > t\right) = \Pr\left(\exp(\lambda T) > \exp(\lambda t)\right) \le \exp(-\lambda t) E\left[\exp(\lambda T)\right] = \exp(-\lambda t) E\left[\exp(\lambda \sum_{i=1}^{A} w_i T_i)\right] \qquad (1)$$

where we have applied Markov inequality (Ross, 2002) to obtain the above inequality. As $\lambda > 0$ is arbitrary, the tightest possible bound is obtained by taking the infimum. **Q.E.D.**

Note that in the special case when $w_a = 1/A$, we have

$$\Pr\left(T > t\right) = \Pr\left(\sum_{i=1}^{A} T_i > tA\right) \le \exp(-\lambda t) E\left[\exp\left(\frac{\lambda}{A} \sum_{i=1}^{A} T_i\right)\right]$$

which is an upper bound on the right-tail probability of the familiar TSTT. Hence, in this special case all bounds in this paper can be thought of as bounds on the unreliability of the conventional TSTT. We also want to emphasize that due to the minimization operation, the upper bound in Lemma 1 is always at least as tight as when the conventional Markov inequality is employed (which corresponds to the special case of $\lambda = 1$).

Another inequality that will be of fundamental importance in the current paper is due to Madansky (1959). Particularly, this inequality will be useful in bounding the right-hand side of the inequality in Lemma 1 (cf. Section 4).

Lemma 2 (Madansky, 1959) Let $f(.)$ be a convex function and let Y denote a random variable with bounded support, i.e. there exists real numbers a and b such that $a \le Y \le b$ with probability one. Then

$$E\left[f(Y)\right] \le \frac{b - E[Y]}{b - a} f(a) + \frac{E[Y] - a}{b - a} f(b).$$

The inequalities in Lemmas 1 and 2 consist of the first order moments only. In the numerical case study in Section 7, we will see that the inclusion of higher order moments can substantially improve the bounds. The next lemma is invaluable in the development of these bounds (cf. Section 5, where we will use Lemma 3 to bound the right-hand side of the inequality in Lemma 1). In particular, Lemma 3 enables the incorporation of the *N*-th moment in the bounds (where *N* is a user-specified positive integer).

Lemma 3 Let $\varsigma > 0$ and suppose that X is a non-negative random variable that is bounded from above, i.e. there exists a real number *b* such that $X \le b$ with probability one. Then

$$E\left[\exp(\varsigma X)\right] \le \sum_{k=0}^{N-1} \frac{\varsigma^k E[X^k]}{k!} + \frac{E[X^N]}{b^N}\left(\exp(\varsigma b) - \sum_{k=0}^{N-1} \frac{\varsigma^k b^k}{k!}\right).$$

Proof Using the Taylor series expansion of the exponential function and the linearity property of the mathematical expectation, we have

$$E\left[\exp(\varsigma X)\right] = E\left[\sum_{k=0}^{\infty} \frac{(\varsigma X)^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{E[(\varsigma X)^k]}{k!} = \sum_{k=0}^{N-1} \frac{E[(\varsigma X)^k]}{k!} + \sum_{k=N}^{\infty} \frac{\varsigma^k E[X^N X^{k-N}]}{k!}.$$

Since $X \le b$ with probability one, we have $X^N X^{k-N} \le X^N b^{k-N}$ with probability one. Hence,

$$E[X^N X^{k-N}] \le E[X^N b^{k-N}] = b^{k-N} E[X^N]$$

so that

$$\sum_{k=N}^{\infty} \frac{\varsigma^k E[X^N X^{k-N}]}{k!} \le \sum_{k=N}^{\infty} \frac{\varsigma^k b^{k-N} E[X^N]}{k!} = E[X^N]\sum_{k=N}^{\infty} \frac{\varsigma^k b^{k-N}}{k!} = \frac{E[X^N]}{b^N}\left(\exp(\varsigma b) - \sum_{k=0}^{N-1} \frac{\varsigma^k b^k}{k!}\right)$$

which completes the proof. **Q.E.D.**

Two immediately corollaries are:

Corollary 1 ($N = 1$) Let $\varsigma > 0$ and suppose that X is a non-negative random variable that is bounded from above, i.e. there exists a real number *b* such that $X \le b$ with probability one. Then

$$E\left[\exp(\varsigma X)\right] \le 1 + \frac{E[X]}{b}\left(\exp(\varsigma b) - 1\right).$$

Corollary 2 ($N = 2$) Let $\varsigma > 0$ and suppose that X is a non-negative random variable that is bounded from above, i.e. there exists a real number *b* such that $X \le b$ with probability one. Then

$$E\left[\exp(\varsigma X)\right] \le 1 + \varsigma E[X] + \frac{E[X^2]}{b^2}\left(\exp(\varsigma b) - 1 - \varsigma b\right).$$

4. An Upper Bound using the First Order Moments

In this section we derive an upper bound on the unreliability function of *T*. Contrary to existing research in this area (e.g. Chen et al., 2002; Sumalee and Watling, 2008), we do not require that the probability distributions are known. We only assume that the mean travel times are given and that the travel times lie within known and finite intervals. The main result of this section is summarized in Proposition 1.

Proposition 1 (Upper bound using the first order moment) Suppose that $t_{al} \le T_a \le t_{au}$ with probability one. Moreover, assume that T_a are independent, then

$$\Pr\left(T > t\right) \le \inf_{\lambda > 0}\left[\exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(\frac{E[T_a](\exp(\lambda w_a t_{au}) - \exp(\lambda w_a t_{al}))}{t_{au} - t_{al}} + \frac{t_{au}\exp(\lambda w_a t_{al})}{t_{au} - t_{al}} - \frac{t_{al}\exp(\lambda w_a t_{au})}{t_{au} - t_{al}}\right)\right)^A\right].$$

Proof By independence and Madansky inequality, (1) reduces to

$$\Pr\left(T>t\right)\le \exp(-\lambda t)\prod_{i=1}^{A} E\left[\exp(\lambda w_i T_i)\right]$$
$$\le \exp(-\lambda t)\prod_{a=1}^{A}\left(\frac{t_{au}-E[T_a]}{t_{au}-t_{al}}\exp(\lambda w_a t_{al})+\frac{E[T_a]-t_{al}}{t_{au}-t_{al}}\exp(\lambda w_a t_{au})\right).$$

Applying the arithmetic mean- geometric mean (AM-GM) inequality we have

$$\left(\prod_{a=1}^{A}\left(\frac{t_{au}-E[T_a]}{t_{au}-t_{al}}\exp(\lambda w_a t_{al})+\frac{E[T_a]-t_{al}}{t_{au}-t_{al}}\exp(\lambda w_a t_{au})\right)\right)^{1/A}$$
$$\le \frac{1}{A}\sum_{a=1}^{A}\left(\frac{t_{au}-E[T_a]}{t_{au}-t_{al}}\exp(\lambda w_a t_{al})+\frac{E[T_a]-t_{al}}{t_{au}-t_{al}}\exp(\lambda w_a t_{au})\right)$$
$$=\frac{1}{A}\sum_{a=1}^{A}\left(\frac{E[T_a](\exp(\lambda w_a t_{au})-\exp(\lambda w_a t_{al}))}{t_{au}-t_{al}}+\frac{t_{au}\exp(\lambda w_a t_{al})}{t_{au}-t_{al}}-\frac{t_{al}\exp(\lambda w_a t_{au})}{t_{au}-t_{al}}\right).$$

Hence

$$\Pr\left(T>t\right)\le \exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(\frac{E[T_a](\exp(\lambda w_a t_{au})-\exp(\lambda w_a t_{al}))}{t_{au}-t_{al}}+\frac{t_{au}\exp(\lambda w_a t_{al})}{t_{au}-t_{al}}-\frac{t_{al}\exp(\lambda w_a t_{au})}{t_{au}-t_{al}}\right)\right)^{A} \quad (2)$$

Since $\lambda > 0$ is arbitrary, the result follows. **Q.E.D.**

We conclude this section with two remarks. First, one has no guarantee that the above derived bound (and the bounds in the next section) always assumes values in the natural interval [0, 1]. However, to follow standard practice – for instance, nothing guarantees that Markov inequality (Ross, 2002) yields bounds valued between 0 and 1 (it sometimes does not) – we have chosen to state the bounds as above. While it is not possible to guarantee that the bound in Proposition 1 (and in other propositions in this paper) does not exceed unity, it is possible to guarantee that the above upper bound (and the upper bounds found in Propositions 2-4) is always larger or equal to zero. This can be easily seen by noting that the result in Proposition 1 (Propositions 2-4) was derived by finding upper bounds on the right-hand side of (1), which is always non-negative due to the exponential function (which ensures that the infimum always exists). In numerical studies, one can always adjust the bounds to ensure that they lie in the natural interval (see Section 7). Second, in traditional studies on transportation system reliability assessment, one could further bound the right-hand side of (2). For example, in Ng et al. (2010b) where capacities were assumed to be uncertain and where the BPR function was adopted, the right-hand side of (2) has been further bounded in terms of the mean capacities. Apart from the availability of these moment data (in particular, when higher order moments are to be included, see Section 5), the resulting bounds become less tight, which was part of our motivation to work with travel times directly in the current paper.

5. Bounds using the First *N* Moments

In the previous section, we have derived bounds that were based on the first order moments only. As such, they might not be as tight as one wants in practice. In this section we develop a more general framework to develop bounds that potentially can incorporate moments of order up to *N*, provided that they exist. A careful inspection of the proof of Proposition 1 reveals that the fundamental reason that higher order moments cannot be incorporated into the bounds developed in Section 4 is that Madansky inequality (cf. Lemma 2) do not account for these higher order moments. Hence, a fundamentally different approach is needed. Lemma 3 forms the foundation for this new approach. Since means and variances are the most widely used moments in practice, we have explicitly stated the results for the cases $N \le 2$ (see Propositions 2 and 3 below). The general result where the first *N* moments are available is given in Proposition 4.

In order to illustrate the importance of the inclusion of higher order moments, we start with an alternative upper bound that only relies on mean values (cf. Proposition 1).

Proposition 2 (Alternative upper bound using the first order moment) Suppose that $T_a \le t_{au}$ with probability one. Moreover, assume that T_a are independent, then

$$\Pr\left(T>t\right)\le\inf_{\lambda>0}\left[\exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(1+\frac{E[T_a]}{t_{au}}\left(\exp(\lambda w_a t_{au})-1\right)\right)\right)^{A}\right].$$

Proof By independence and Corollary 1, (1) reduces to

$$\Pr\left(T>t\right)\le\exp(-\lambda t)\prod\nolimits_{i=1}^{A}E\left[\exp(\lambda w_i T_i)\right]\le\exp(-\lambda t)\prod\nolimits_{a=1}^{A}\left(1+\frac{E[T_a]}{t_{au}}\left(\exp(\lambda w_a t_{au})-1\right)\right).$$

As in the proof of Proposition 1, the AM-GM inequality yields

$$\left(\prod\nolimits_{a=1}^{A}\left(1+\frac{E[T_a]}{t_{au}}\left(\exp(\lambda w_a t_{au})-1\right)\right)\right)^{1/A}\le\frac{1}{A}\sum\nolimits_{a=1}^{A}\left(1+\frac{E[T_a]}{t_{au}}\left(\exp(\lambda w_a t_{au})-1\right)\right).$$

Hence

$$\Pr\left(T>t\right)\le\exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(1+\frac{E[T_a]}{t_{au}}\left(\exp(\lambda w_a t_{au})-1\right)\right)\right)^{A}.$$

Since $\lambda>0$ is arbitrary, the result follows. **Q.E.D.**

The inclusion of variance information yields Proposition 3.

Proposition 3 (Upper bound using the first and second order moments) Suppose that $T_a\le t_{au}$ with probability one. Moreover, assume that T_a are independent, then

$$\Pr\left(T>t\right)\le\inf_{\lambda>0}\left[\exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(1+\lambda w_a E[T_a]+\frac{E[T_a^2]}{t_{au}^2}\left(\exp(\lambda w_a t_{au})-1-\lambda w_a t_{au}\right)\right)\right)^{A}\right].$$

Proof By independence and Corollary 2, (1) reduces to

$$\Pr\left(T>t\right)\le\exp(-\lambda t)\prod\nolimits_{i=1}^{A}E\left[\exp(\lambda w_i T_i)\right]\le\exp(-\lambda t)\prod\nolimits_{a=1}^{A}\left(1+\lambda w_a E[T_a]+\frac{E[T_a^2]}{t_{au}^2}\left(\exp(\lambda w_a t_{au})-1-\lambda w_a t_{au}\right)\right).$$

As in the proof of Proposition 1, the AM-GM inequality yields

$$\left(\prod\nolimits_{a=1}^{A}\left(1+\lambda w_a E[T_a]+\frac{E[T_a^2]}{t_{au}^2}\left(\exp(\lambda w_a t_{au})-1-\lambda w_a t_{au}\right)\right)\right)^{1/A}$$
$$\le\frac{1}{A}\sum\nolimits_{a=1}^{A}\left(1+\lambda w_a E[T_a]+\frac{E[T_a^2]}{t_{au}^2}\left(\exp(\lambda w_a t_{au})-1-\lambda w_a t_{au}\right)\right).$$

Hence

$$\Pr\left(T>t\right)\le\exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(1+\lambda w_a E[T_a]+\frac{E[T_a^2]}{t_{au}^2}\left(\exp(\lambda w_a t_{au})-1-\lambda w_a t_{au}\right)\right)\right)^{A}.$$

Since $\lambda>0$ is arbitrary, the result follows. **Q.E.D.**

When the first N moments are known, we have the general result in Proposition 4.

Proposition 4 (Upper bound using the first *N* moments) Suppose that $T_a\le t_{au}$ with probability one. Moreover, assume that T_a are independent, then

$$\Pr\left(T>t\right)\le\inf_{\lambda>0}\left[\exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(\sum_{k=0}^{N-1}\frac{\left(\lambda w_a\right)^k E[T_a^k]}{k!}+\frac{E[T_a^N]}{t_{au}^N}\left(\exp(\lambda w_a t_{au})-\sum\nolimits_{k=0}^{N-1}\frac{\left(\lambda w_a\right)^k t_{au}^N}{k!}\right)\right)\right)^{A}\right].$$

Proof By independence and Lemma 3, (1) reduces to

$$\Pr\left(T>t\right)\le \exp(-\lambda t)\prod_{a=1}^{A}\left(\sum_{k=0}^{N-1}\frac{\left(\lambda w_a\right)^k E[T_a^k]}{k!}+\frac{E[T_a^N]}{t_{au}^N}\left(\exp(\lambda w_a t_{au})-\sum_{k=0}^{N-1}\frac{\left(\lambda w_a\right)^k t_{au}^N}{k!}\right)\right).$$

Using the AM-GM inequality, the above inequality can be expressed as

$$\Pr\left(T>t\right)\le \exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(\sum_{k=0}^{N-1}\frac{\left(\lambda w_a\right)^k E[T_a^k]}{k!}+\frac{E[T_a^N]}{t_{au}^N}\left(\exp(\lambda w_a t_{au})-\sum_{k=0}^{N-1}\frac{\left(\lambda w_a\right)^k t_{au}^N}{k!}\right)\right)\right)^A .$$

Since $\lambda > 0$ is arbitrary, the result follows. **Q.E.D.**

Five remarks are in order. First, a sufficient condition for the Central Limit Theorem to hold, i.e. for T to converge in distribution to some normal random variable – note that in such a case the development of bounds on the unreliability function would be simplified – is that T_a are bounded and $\sum_a Var[T_a] \to \infty$ (Billingsley, 1995). Here we have used Var[X] to denote the variance of the random variable X. The former assumption is, by definition, satisfied in the current paper; the assumption that $\sum_a Var[T_a] \to \infty$ is a much stronger assumption that we do not impose in this paper (cf. Lo et al., 2006). Consequently, our results hold in much more generality. Second, note that the bound in Proposition 1 is an explicit function of the left end point t_{al} of the support of T_a , in Propositions 2 to 4 the bounds only depend on the right end point t_{au} of the support. From their proofs, it is clear that T_a can be bounded from below too, i.e. there can exist a value $t_{al} > 0$ such that $t_{al} \le T_a$ with probability one. But since the bound will not involve this lower end point, we have chosen not to explicitly state this bounded-from-below-assumption in the propositions. Third, notice that we have not explicitly stated the assumption that the moments are finite since the boundedness assumption on T_a implies that all its moments exist. Fourth, in Clark and Watling (2005), the first four moments alone determined the unreliability of travel time, whereas N can be much larger here (note that they considered demand uncertainty). Of course, the critical difference is that our approach is distribution-free while Clark and Watling assumed a very specific (i.e. the multivariate normal) distribution to arrive at their very specific probability value. Fifth, bounds using higher order moments generally require more input data (one needs to specify these moments), which might not be easy (e.g. what is the 10-th order moment of the link travel time?). However, when travel time data are available, the estimation of the higher order moments are as easy as the estimation of the lower order moments (cf. the method of moments in statistics, Casella and Berger, 2002; also see the discussion in Section 2).

6. Some Proofs of Convexity

The bounds derived in the previous sections all involve the minimization of some univariate objective function in λ. Clearly, any feasible choice of λ results in a valid bound, with different values for λ resulting in different bounds. In particular, one can set $\lambda = 1$, so that the standard Markov inequality is obtained in (1). By solving the minimization problems, tighter bounds can be found (which is of course dependent on how the optimization is performed – in this paper we use MATLAB's "fminbnd" routine). To ensure global optimality, convexity of the functions to be minimized is desired. Because of the wide-spread use of first and second order moments, in this section we prove that the functions in Propositions 1 and 2 are indeed convex via second order characterizations (and some tedious algebra). Unfortunately, we have not been able to establish convexity of the function in Proposition 3. We shall only provide some visual analysis of this function in the next section. In any case (i.e. convex or not), we want to emphasize that the bounds are valid for *any* feasible choice of λ. Moreover, as will be seen in Section 7, irrespective of whether the functions are convex, the bounds are very useful. In other words, convexity is a luxury, not a necessity, in the current paper. The following two results are of crucial importance in establishing convexity in the current section (Boyd and Vandenberghe, 2004).

Lemma 4 If f is convex and non-decreasing and if h is convex, then $g(\lambda) = f(h(\lambda))$ is also convex.

Lemma 5 Let H_i , i =1,2,…A be convex functions. Moreover, let γ_i be non-negative real numbers, then $\sum_i \gamma_i H_i$ is also convex.

Proposition 5 The function

$$g_1(\lambda) \equiv \exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(\frac{E[T_a](\exp(\lambda w_a t_{au}) - \exp(\lambda w_a t_{al}))}{t_{au} - t_{al}} + \frac{t_{au}\exp(\lambda w_a t_{al})}{t_{au} - t_{al}} - \frac{t_{al}\exp(\lambda w_a t_{au})}{t_{au} - t_{al}}\right)\right)^A$$

in Proposition 1 is convex.

Proof Note that we can rewrite $g_1(\lambda)$ as

$$g_1(\lambda) = f_1(h_1(\lambda))$$

where

$$f_1(x) = x^A$$

and

$$h_1(\lambda) = \exp\left(\frac{-\lambda t}{A}\right)\frac{1}{A}\sum_{a=1}^{A}\left(\frac{E[T_a](\exp(\lambda w_a t_{au}) - \exp(\lambda w_a t_{al}))}{t_{au} - t_{al}} + \frac{t_{au}\exp(\lambda w_a t_{al})}{t_{au} - t_{al}} - \frac{t_{al}\exp(\lambda w_a t_{au})}{t_{au} - t_{al}}\right).$$

Thus if we can show that $h_1(\lambda)$ is convex, we have proved the claim as $f_1(x)$ is clearly convex and non-decreasing for $x \geq 0, A \geq 1$. Indeed, $h_1(\lambda)$ is convex since $t_{al} \leq E[T_a] \leq t_{au}$ and

$$\frac{d^2}{d\lambda^2}\exp\left(\frac{-\lambda t}{A}\right)\left(\frac{E[T_a](\exp(\lambda w_a t_{au}) - \exp(\lambda w_a t_{al}))}{t_{au} - t_{al}} + \frac{t_{au}\exp(\lambda w_a t_{al})}{t_{au} - t_{al}} - \frac{t_{al}\exp(\lambda w_a t_{au})}{t_{au} - t_{al}}\right)$$

$$= \exp\left(\frac{-\lambda t}{A}\right)\frac{1}{A^2}\frac{((t - Aw_a t_{au})^2 (E[T_a] - t_{al})\exp(\lambda w_a t_{au}) - (t - Aw_a t_{al})^2 (E[T_a] - t_{au})\exp(\lambda w_a t_{al}))}{t_{au} - t_{al}} \geq 0.$$

By Lemma 5, the result follows. **Q.E.D.**

With the aid of a computer algebra package, one can establish the next convexity result.

Proposition 6 The function

$$g_2(\lambda) \equiv \exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(1 + \frac{E[T_a]}{t_{au}}\left(\exp(\lambda w_a t_{au}) - 1\right)\right)\right)^A$$

in Proposition 2 is convex.

Proof As above, we can rewrite $g_2(\lambda)$ as

$$g_2(\lambda) = f_2(h_2(\lambda))$$

where

$$f_2(x) = x^A$$

and

$$h_2(\lambda) = \exp\left(\frac{-\lambda t}{A}\right)\frac{1}{A}\sum_{a=1}^{A}\left(1 + \frac{E[T_a]}{t_{au}}\left(\exp(\lambda w_a t_{au}) - 1\right)\right).$$

If $h_2(\lambda)$ is convex, the claim follows directly from Lemma 4. To prove convexity of $h_2(\lambda)$, by Lemma 5, it suffices to show that

$$\frac{d^2}{d\lambda^2}\exp\left(\frac{-\lambda t}{A}\right)\left(1 + \frac{E[T_a]}{t_{au}}\left(\exp(\lambda w_a t_{au}) - 1\right)\right) \geq 0 .$$

With some tedious algebra, one can show that

$$\frac{d^2}{d\lambda^2}\exp\left(\frac{-\lambda t}{A}\right)\left(1 + \frac{E[T_a]}{t_{au}}\left(\exp(\lambda w_a t_{au}) - 1\right)\right) = \frac{1}{A^2 t_{au}}\exp\left(\frac{-\lambda t}{A}\right)(at^2 + bt + c)$$

where

$$\begin{aligned} a &= t_{au} + E[T_a]\left(\exp(\lambda w_a t_{au}) - 1\right) \\ b &= -2Aw_a t_{au} E[T_a] \\ c &= A^2 w_a^2 t_{au}^2 E[T_a] \exp(\lambda w_a t_{au}). \end{aligned}$$

One can verify that the discriminant of the quadratic function $at^2 + bt + c$ is given by

$$4E[T_a] w_a^2 t_{au}^2 A^2 \exp(\lambda w_a t_{au})(E[T_a] - t_{au})$$

which is non-positive since $E[T_a] \le t_{au}$. This implies that $at^2 + bt + c \ge 0$, completing the proof. **Q.E.D.**

7. Numerical Demonstration

In this section we use the well-known Sioux Falls network (Figure 1) with $A = 76$ to illustrate the bounds derived in the previous sections. Given the wide-spread use of first and second order moments in the transportation profession, we shall focus our attention on the cases $N \le 2$ only. The network data reported in Suwansirikul et al. (1987) provide the starting point of our numerical analysis. Without loss of generality, we solved an instance of the deterministic user equilibrium assignment (Sheffi, 1985) based on the data in Suwansirikul et al. (1987). The resulting total travel times on the links are assumed to be the expected values $E[T_a]$. Upper and lower bounds on the travel times are obtained by the multiplication of their mean values with factors q_{al} and q_{au} where $0 < q_{al} < 1 < q_{au}$, i.e. $t_{al} = q_{al} E[T_a]$ and $t_{au} = q_{au} E[T_a]$. Likewise, the second moments are obtained by the multiplication of $\left(E[T_a]\right)^2$ with $c \ge 1$ (since $0 \le Var[T_a] = E[T_a^2] - \left(E[T_a]\right)^2$) such that $c\left(E[T_a]\right)^2 \le (t_{au} - t_{al})^2 / 4 + (E[T_a])^2$. This bound on the largest possible second moment can, for example, be found in Theorem 4.1 of Seaman and Odell (1985). It follows directly from the fact that a Bernoulli random variable (Ross, 2002) has the highest possible variance among all random variables on the interval [0,1].

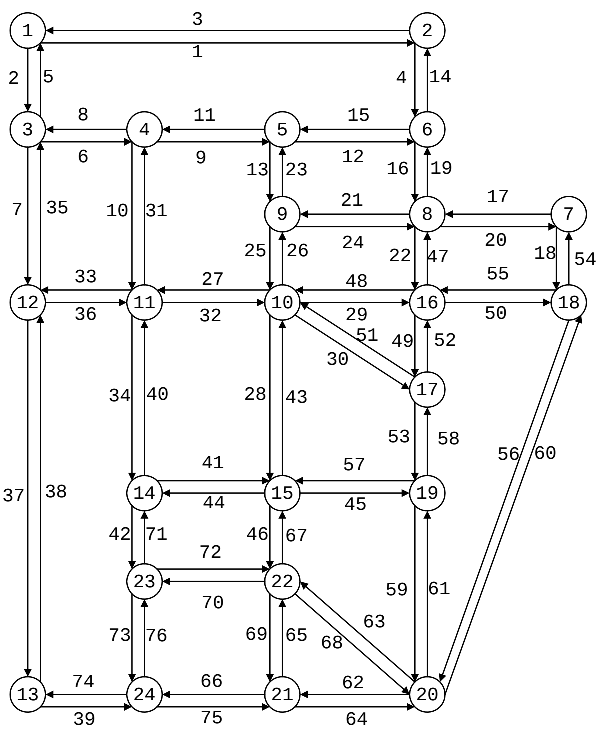

Figure 1: Sioux Falls test network with node and link numbers.

To evaluate the bounds derived in Propositions 1 through 3, we employ the “fminbnd” routine available in MATLAB® which is an optimization procedure based on golden section search and parabolic interpolation (Press et al., 2007). The interval in which to search for an optimal value for λ was constrained to be $(0,50]$ that was empirically found to contain a minimizer for the cases considered. Computational tests demonstrated the effectiveness and computational efficiency of this optimization technique (e.g. Figure 2 was obtained within 1 second on an HP laptop computer with a 4.00 GHz AMD Turion™ Dual-Core Processor and 4 GB RAM). Note that from the convexity results in Section 6 it is known that the resulting bounds in Propositions 1 and 2 are global minima, whereas it is not guaranteed that the bound from Proposition 3 is minimized with respect to λ. Some empirical evidence that suggests that they might be global minima will be presented at the end of this section.

In this case study, we examine the conventional TSTT (Ng and Waller, 2010), i.e. we set $w_a = 1/A$ (see the discussion right below Lemma 1). Unless stated otherwise, we set $q_{al} = 0.2$ and $q_{au} = 3$ in the following numerical experiments. Figure 2 depicts the upper bounds from Propositions 1 through 3 for the case $c = 1.1$ as a function of the TSTT (using the demand values reported in Suwansirikul et al., 1987). While it was not necessary in our case study, one can always adjust the predicted bounds so that the resulting value lies between 0 and 1 (cf. the remark at the end of Section 4). That is, suppose that the *predicted* upper bound (i.e. the theoretical bounds derived in Sections 4 and 5) is given by z_u, then the adjusted upper bound would be given by $\min\{1, z_u\}$.

From Figure 2, we can make several interesting observations. A theoretical upper bound on the TSTT is given by the TSTT associated with the case when $T_a = q_{au}E[T_a]$, henceforth referred to as the worst case TSTT (Note that the worst case TSTT is an bound on the TSTT. They are not to be confused with the bounds derived in Sections 4 and 5 that were bounds on the *right-tail probability* of the TSTT). The worst case TSTT for the instance examined in Figure 2 is equal to 302 hours. All upper bounds are able to capture this fact very accurately: they predict a probability of zero that the worst case TSTT is exceeded. In fact, they predict that a much larger range of TSTT values are not possible (see Figure 2). From the figure, we can also see that in this specific instance, the bound in Proposition 3 is tighter than the bound in Proposition 1 which is turn is tighter than the bound in Proposition 2. Since the bounds in Proposition 1 and Proposition 3 are developed using fundamentally different approaches (e.g. in Proposition 1 we relied on Madansky inequality and in Proposition 3 we did not), it is not possible to assess the importance of the inclusion of variance information based on these two bounds. To do this, we need to consider the bounds from Propositions 2 and 3. From Figure 2 it is clear that the incorporation of variance information results in substantially tighter bounds in this case. For example, for TSTT = 130 hours, the bound from Proposition 2 predicts that this TSTT value is exceeded with a probability of at most 0.43, whereas, the bound from Proposition 3 tells us that the probability is at most 0.09 (which is a reduction in uncertainty of about 80%). Finally, while not really relevant from a reliability perspective (since we are mostly interested in the likelihood that relatively high TSTT values are exceed), we note that for TSTT values less than about 102 hours, the bounds are trivial (the mean TSTT in this case was 101 hours, i.e. $\sum E[T_a] = 101$).

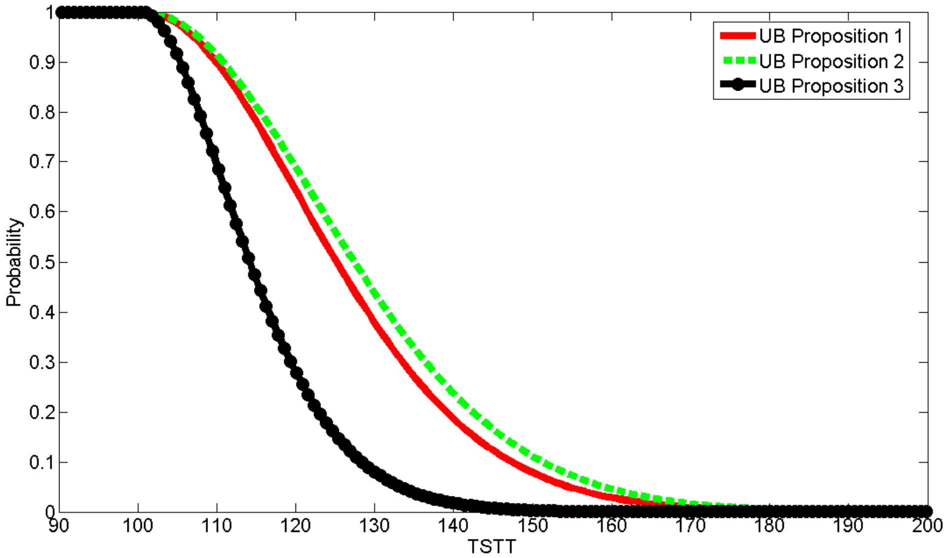

Figure 2: Upper bounds on the unreliability function for $c = 1.1$.

Before we proceed to the next numerical experiment, let us first take a step back and examine in a thought experiment under what circumstances variance/ second order moment information is likely to result in tighter bounds: Clearly, the higher the variance, the less information it actually gives us. Hence it seems to be logical that when the variance increases (due to an increase in the second order moment), *ceteris paribus*, the difference between the bounds from Propositions 2 and 3 will get smaller. Figure 3 ($c = 2.2$) confirms this reasoning. It shows that the bound from Proposition 3 has moved much closer to the bound from Proposition 1 (which obviously has not changed as it does not depend on the second order moment). In the limiting case when $E[T_a^2] = (t_{au} - t_{al})^2 / 4 + (E[T_a])^2$ (cf. Seaman and Odell, 1985), the bound in Proposition 3 becomes less tight than its counterpart from Proposition 1, see Figure 4. (Now it should be clear that it is impossible to give a general rule of thumb for selecting the value of *N* as this would depend on how "informative" the moments are that one considers to include in the analysis. Of course, other issues such as the availability of travel time data, i.e. how easy it is to estimate higher order moments, are critical too.) At first, it might seem contradictory that a bound that contains variance information is less tight than one that does not. This is perfectly fine since the bounds were derived using very different approaches. Moreover, the bound from Proposition 3 almost coincides with the bound in Proposition 2. Note that we have not given any theoretical guarantee on the ordering of the bounds. This would require a discussion of the sharpness of the bounds, i.e. whether there exists random travel times such that the bounds are attained, which we have left as an important future research topic. In light of these observations, for now, we suggest to use the minimum of the upper bounds as *the* bound in reliability studies – note that this approach is particularly feasible since each bound can be obtained within a fraction of a second.

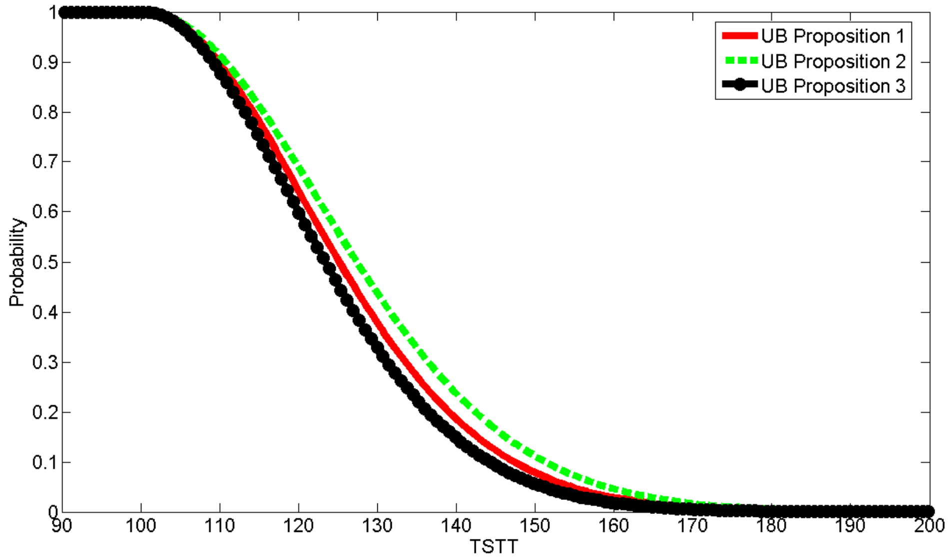

Figure 3: Upper bounds on the unreliability function for $c = 2.2$ **.**

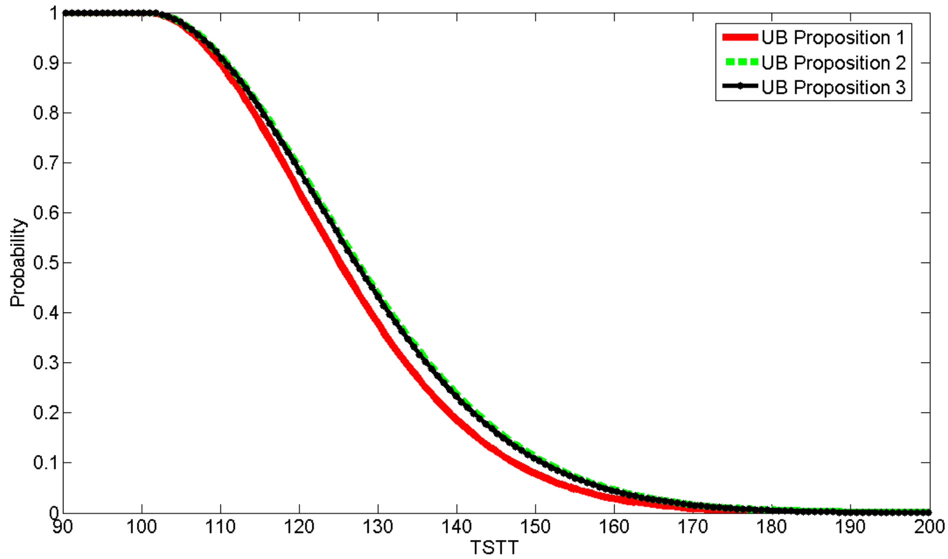

Figure 4: Upper bounds on the unreliability function under maximum variance.

In Figure 5 we have shown the effect of changing the right end points of the support of the travel times (as captured by the factor q_{au}). For the reader's convenience, we have repeated Figure 2 in the upper figure of Figure 5. When the right end point of the support becomes larger ($q_{au} = 6$), *ceteris paribus*, the bounds become wider as higher travel times become possible (middle of Figure 5). On the other hand, when the right end point of the support gets smaller ($q_{au} = 1.5$), the bounds tighten. While all bounds are dependent on q_{au}, as noted above, not all bounds (cf. Propositions 2 and 3) are a function of the left end points of the support (as captured by the factor q_{al}). Hence, varying q_{al} will not have any impact on these bounds, as is confirmed by Figure 6. The only bound that changes is the bound from Proposition 1 that tightens when q_{al} is increased (which corresponds to a smaller range of travel times being possible, see middle of Figure 6) and widens when is q_{al} is decreased (which corresponds to a larger range of travel times being possible, see bottom of Figure 6). Note that the bound virtually coincides with the bound derived in Proposition 2 when q_{al} is small.

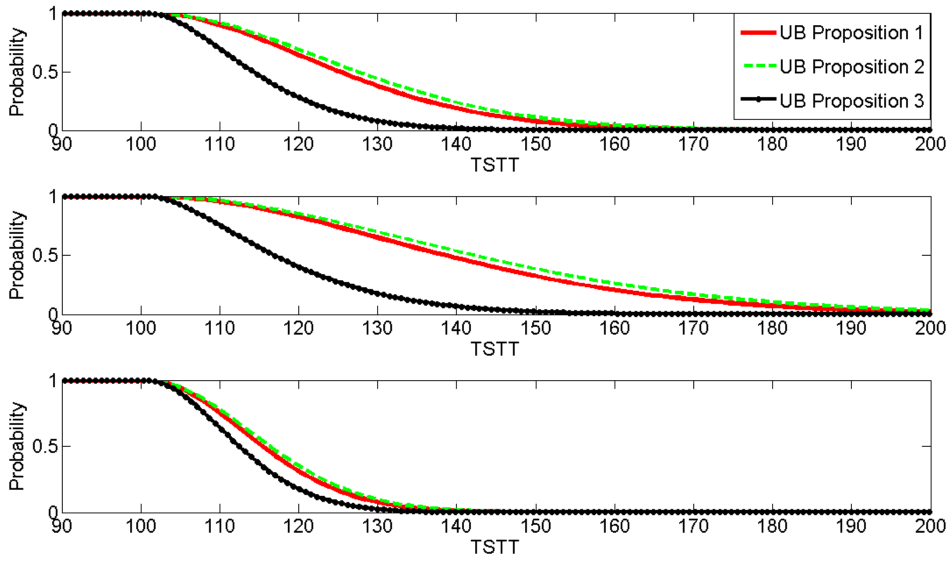

Figure 5: Upper bounds as a function of q_{au}. Upper: q_{au} = 3, middle: q_{au} = 6, lower: q_{au} = 1.5.

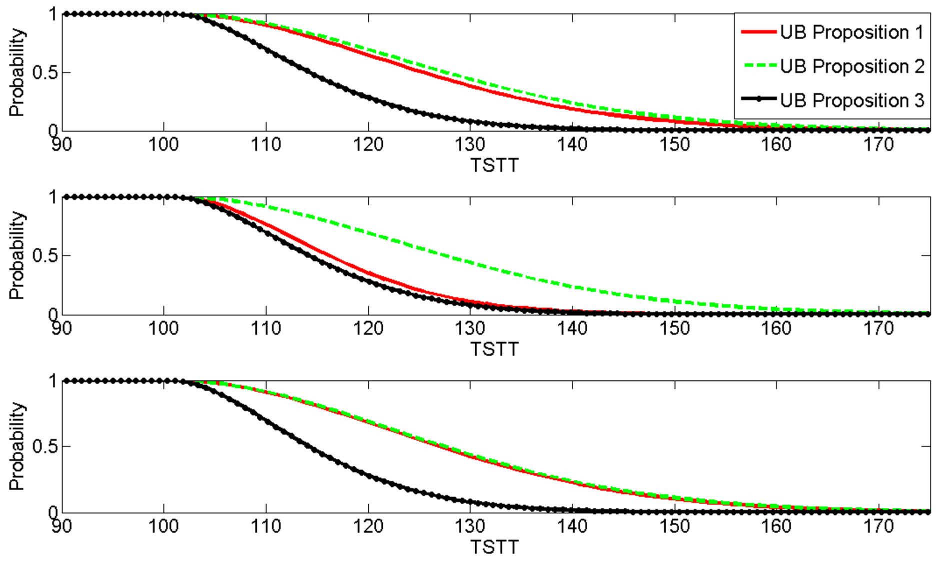

Figure 6: Upper bounds as a function of q_{al}. Upper: q_{al} = 0.2, middle: q_{al} = 0.8, lower: q_{al} = 0.05.

We want to emphasize that our bounds are distribution-free. Hence, we believe that any (simulation) exercise to evaluate the bounds based on specific distributional assumptions would be misleading since the quality of the bounds can be made better or worse, depending on the choice of the underlying probability distributions. (Note that if distributions were known, one would not adopt the proposed distribution-free approach to begin with; the Fourier transform-based approach proposed in Ng and Waller (2010) would then be more appropriate, for example.) However, for a *very* rough idea of how the bounds compare to specific distributions, we have used Monte Carlo simulation to estimate a number of points of the unreliability function based various probability distributions (assuming c = 1.1, $q_{al} = 0.2$ and $q_{au} = 5$), see Figure 7. Particularly, we generated 10000 independent realizations of the link travel times based on the uniform, normal and gamma distributions under consideration (with MATLAB's random number generator), using the same mean and variance as the instance in Figure 2, and counted the number of times k the sample TSTT exceeded a given value for t. The fraction k/10000 was then used as the point estimate of the unreliability at t, see Figure 7. To emphasize the point that we only require independence (i.e. we do not require that all link travel times come from the same parametric family of probability distributions – as was the case in the above simulations), we have also estimated an unreliability curve where the first 38 links have normal distributions and the second 38 links have gamma distributions. In Figure 7 we have indicated this as the normal/gamma mixture distribution. From the figure, it is clear that our conjecture is confirmed: the closeness to the bound is a function of the probability distribution chosen. For example, the unreliability curve based on the normal distribution lies much closer to the distribution-free bound than the unreliability function estimated based on the gamma distribution. Thus, again, the reader should keep in mind that Figure 7 only gives a *very* rough idea of the performance of the distribution-free bounds. Interestingly, while the distance between the distribution-free bound and the Monte-Carlo-based unreliability curves can be large, results in Ng (2010) suggest that the former can be used as a computationally efficient proxy for the latter in network design problems aiming to improve the reliability of transportation systems (e.g. Chen et al., 2009).

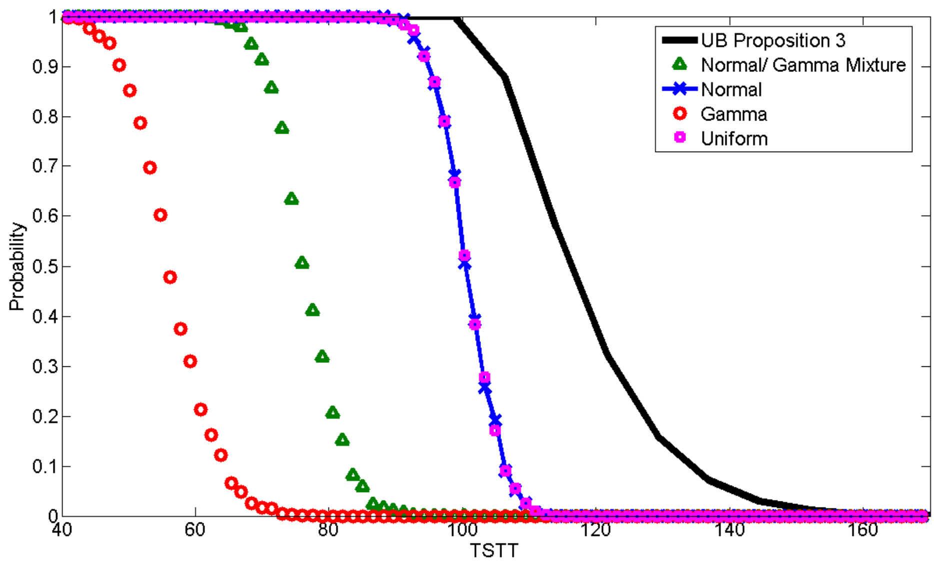

Figure 7: Monte Carlo estimates of the unreliability function.

Before we conclude our numerical case study, we present some graphs of the function

$$g_3(\lambda) \equiv \exp(-\lambda t)\left(\frac{1}{A}\sum_{a=1}^{A}\left(\exp\left(\lambda w_a E[T_a] + \frac{E[T_a^2]}{t_{au}^2}\left(\exp(\lambda w_a t_{au}) - 1 - \lambda w_a t_{au}\right)\right)\right)\right)^A \tag{3}$$

that appeared in Proposition 3. Figure 8 shows three (t = 1.5, 1.6 and 1.7 which corresponds to TSTT = 114, 122 and 129 hours, respectively) representative graphs of the function in (3) using the Sioux Falls network data and c = 1.1 , suggesting that the functions are nicely convex. Of course, we are fully aware that these observations, by no means, constitute a proof of convexity. In future work we hope to resolve this and the general issue of convexity (cf. Proposition 4) more rigorously. But as indicated, convex or not, the bounds are valid for *any* feasible choice of λ . Given their demonstrated usefulness, we believe that convexity is a pure luxury, not a necessity in the current paper.

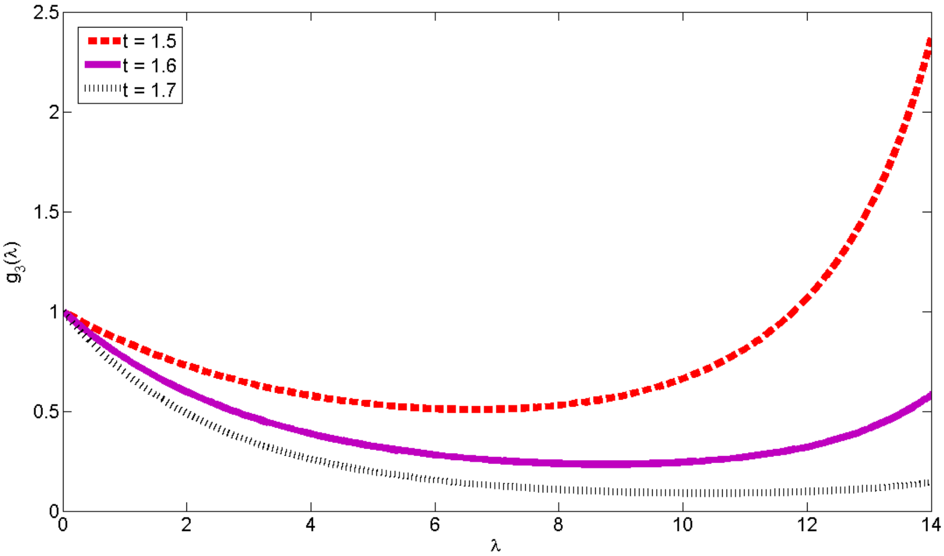

Figure 8: Three typical convex graphs of $g_3(\lambda)$.

8. Conclusions

A common assumption in the current literature on the reliability assessment of stochastic transportation systems is that the exact probability distributions of the sources of uncertainty are known explicitly. However, this distribution may be unavailable (inaccurate) in reality because we may have no (insufficient) data to calibrate the distribution. In this paper we relax this rather restrictive assumption and present a travel time reliability assessment technique based on probability inequalities. Instead of the specification of the probability distributions, the methodology only requires the specification of moments (up to order N) and a set of bounded intervals in which the random quantities are expected to reside. The price to pay for this relaxation is that we only obtain bounds on the unreliability function as opposed to exact probabilities of events (that are only as good as the underlying hypothetical assumptions on the probability distributions). The gain of such an approach is computational efficiency: bounds can be obtained within a fraction of a second. We also depart from previous modeling paradigms in that we directly work with the travel time, rather than for example, with road capacities (we argued that it is much easier to answer the question what the variance of travel time is than what the variance of road capacity is). The only assumption is that the travel times are independent across links (e.g., Lo and Tung, 2003; Lo et al., 2006, Ng and Waller, 2010).

A numerical case study using the well-known Sioux Falls test network revealed a number of important properties of the bounds. First, the bounds were found to be nontrivial for the most interesting (in terms of reliability assessment) subset of the feasible region, i.e. the higher travel times. Second, first order moment-based bound can potentially be significantly improved by the inclusion of higher order moments. The improvement is larger when the additional information is more informative. For example, a smaller second order moment (i.e. smaller variance) is more informative than larger values (i.e. larger variances). Third, the proposed methodology is extremely computationally efficient, in fact, the most efficient among all transportation reliability assessment techniques proposed to date.

The introduction of the distribution-free paradigm is without doubt a major contribution of the current paper to the transportation systems reliability literature. While we have developed bounds that are extremely useful (until now, not much – if anything – could be said about transportation system reliability given only moments and supports of random quantities), there is a substantial amount of work left for future research. For example, the sharpness of the bounds needs to be examined: are there travel times with given moments that attain the bounds? Given the state-of-the-art of analytical dependence modeling in the transportation system reliability assessment literature, the independence assumption in the current paper is not a significant limitation. However, it is straightforward to imagine situations where stochastic dependencies cannot be ignored. Fundamentally new statistical techniques are needed to account for such dependencies. We are currently developing such techniques as part of the first author's ongoing work on transportation reliability assessment, which we hope to report in the near future.

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LIST OF FIGURE CAPTIONS