Time-Invariant Uncertain Systems: a Necessary and Sufficient Condition for Stability and Instability via HPD-QLFs

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Abstract

This paper investigates linear systems with polynomial dependence on time-invariant uncertainties constrained in the simplex via homogeneous parameter-dependent quadratic Lyapunov functions (HPD-QLFs). It is shown that a sufficient condition for establishing whether the system is either stable or unstable can be obtained by solving a generalized eigenvalue problem. Moreover, this condition is also necessary by using a sufficiently large degree of the HPD-QLF.

Keywords: Uncertain system, Stability, Instability, Lyapunov function, SMR.

1 Introduction

Various methods have been proposed for stability of linear systems with time-invariant uncertainty constrained in a polytope. Generally, these methods exploit parameter-dependent Lyapunov functions and LMIs, see e.g. [1] which considers Lyapunov functions with linear dependence, [2] which proposes Lyapunov functions with polynomial dependence, [3] which introduces the class of HPD-QLFs, [4] which proposes a general framework for LMI relaxations, [5] where homogeneous solutions are characterized, [6] which addresses the case of semi-algebraic sets, and [7, 8] where matrix-dilation approaches are considered.

Some of these methods provide necessary and sufficient conditions for robust stability. However, the necessity is achieved for an unknown degree of the polynomials used. This implies that, if the system is unstable, no conclusion can be reached. This paper addresses this problem via HPD-QLFs for in the case of polynomial dependence on the uncertainty. It is shown that a sufficient condition for establishing either stability or instability can be obtained by solving a generalized eigenvalue problem, and that this condition is also necessary by using a sufficiently large degree of the HPD-QLF. The idea behind this condition is to exploit the LMI relaxation introduced in [3] via the square matricial representation (SMR)[1] in order to characterize the instability via the presence of suitable vectors in certain eigenspaces.

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[1]The SMR allows one to establish if a polynomial is a sum of squares (SOS) of polynomials via an LMI, see e.g. [9, 10, 11, 12] and references therein.

Before proceeding, it is worth explaining that the proposed approach differs from [13] which investigates robust $\mathcal{H}_\infty$ performance via eigenvalue problems, from [14] which exploits D/G-scaling in the case of rational dependence on the uncertainty, from [15] which derives a recursive algorithm based on the linear fractional representation, and from [16, 17] which propose a branch-and-bound method in the case of linear dependence on the uncertainty.

2 Preliminaries

Notation: $\mathbb{R}, \mathbb{C}$: real and complex numbers; $\mathbb{R}_0$: $\mathbb{R} \setminus \{0\}$; I_n: $n \times n$ identity matrix; $A > 0$: symmetric positive definite matrix; $A \otimes B$: Kronecker's product; A', $\mathrm{tr}(A)$, $\det(A)$: transpose, trace and determinant of A; $\mathrm{vec}(A)$: vector with the columns of A stacked below each other; $\mathrm{spc}(A) = \{\lambda \in \mathbb{C} : \ \det(\lambda I - A) = 0\}$; $\mathrm{span}(v_1, \ldots, v_k) = \{a_1 v_1 + \ldots + a_k v_k, \ a_1, \ldots, a_k \in \mathbb{R}\}$; $\mathrm{sq}(p) = \left(p_1^2, \ldots, p_q^2\right)'$ with $p \in \mathbb{R}^q$; CT, DT: continuous-time and discrete-time; s.t.: subject to.

Let us consider the uncertain system

$$\left\{ \begin{array}{ll} \text{(CT case)} & \dot{x}(t) = A(p)x(t) \\ \text{(DT case)} & x(t+1) = A(p)x(t) \end{array} \right. \quad \forall p \in \mathcal{P} \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $p \in \mathbb{R}^q$ is the uncertain parameter, and $\mathcal{P}$ is the simplex, i.e. $\mathcal{P} = \left\{p \in \mathbb{R}^q : \ \sum_{i=1}^q p_i = 1, \ \ p_i \geq 0\right\}$. The function $A : \mathbb{R}^q \to \mathbb{R}^{n \times n}$ is a *matrix form* of degree m_A, i.e. a matrix whose entries are forms (i.e. homogeneous polynomials) of degree m_A. Let us define

$$\mathcal{A} = \left\{A(p) \in \mathbb{R}^{n \times n} : \ p \in \mathcal{P}\right\}. \tag{2}$$

In the sequel we will say that:

- (CT case) $A(p)$ is stable if and only if $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \mathrm{spc}(A(p))$;
- (DT case) $A(p)$ is stable if and only if $|\lambda| < 1$ for all $\lambda \in \mathrm{spc}(A(p))$;
- $\mathcal{A}$ is stable if and only if $A(p)$ is stable for all $p \in \mathcal{P}$;
- $\mathcal{A}$ (resp., $A(p)$) is unstable if is not stable.

In order to introduce the proposed result, let us define

$$\begin{array}{ll} \text{(CT case)} & d = m_A \\ \text{(DT case)} & d = 2m_A \end{array} \tag{3}$$

and let $p^{\{m\}}$ be a vector containing all monomials of degree m in p. Let us introduce

$$\begin{array}{l} \mathcal{S} = \Big\{S = S' : \ \Delta\left(p^{\{m\}}, S\right) \text{ does not contain} \\ \quad \text{monomials } p_1^{i_1} \ldots p_q^{i_q} \text{ with at least one } i_j \text{ odd}\Big\} \\ \mathcal{U} = \left\{U = U' : \ \Delta\left(p^{\{m+d\}}, U\right) = 0 \ \ \forall p\right\} \end{array} \tag{4}$$

where the notation $\Delta(\cdot,\cdot)$ means

$$\Delta\left(b,B\right)=\left(b\otimes I_n\right)'B\left(b\otimes I_n\right) \tag{5}$$

for any suitable b,B, and let $S(\beta),U(\alpha)$ be linear parameterizations of $\mathcal{S}$,$\mathcal{U}$. Define the functions

$$P(\mathrm{sq}(p),\beta)=\Delta\left(p^{\{m\}},S(\beta)\right) \tag{6}$$

and

$$\begin{array}{l}(\text{CT case})\ Q(\mathrm{sq}(p),\beta)=-B'C-CB\\ (\text{DT case})\ Q(\mathrm{sq}(p),\beta)=\left(\sum_{i=1}^{q}p_i^2\right)^d C-B'CB\end{array} \tag{7}$$

for $B=A(\mathrm{sq}(p))$ and $C=P(\mathrm{sq}(p),\beta)$. Let $R(\beta)$ be a SMR matrix of the matrix form $Q(\mathrm{sq}(p),\beta)$, i.e. a symmetric function satisfying

$$\Delta\left(p^{\{m+d\}},R(\beta)\right)=Q(\mathrm{sq}(p),\beta). \tag{8}$$

The following theorem is given in [3, 18, 12] and investigates robust stability of (1) via a HPD-QLF of degree m, i.e. via a Lyapunov function of the form $x'P(p,\beta)x$.

Theorem 1 *[18] The set $\mathcal{A}$ is stable if and only if there exist m such that the following LMIs hold for some α,β:*

$$\left\{\begin{array}{l}S(\beta)>0\\ R(\beta)+U(\alpha)>0.\end{array}\right. \tag{9}$$

3 Stability and Instability Condition

Let us define

$$T(\beta)=\Delta\left(K,I_{dq}\otimes S(\beta)\right) \tag{10}$$

where K is the matrix satisfying

$$\underbrace{p\otimes\cdots\otimes p}_{d\text{ times}}\otimes p^{\{m\}}=Kp^{\{m+d\}} \tag{11}$$

(see also (18) for a key property of $T(\beta)$) and define

$$\eta^*=\sup_{\alpha,\beta,\eta}\eta\ \text{ s.t. }\left\{\begin{array}{l}S(\beta)>0\\ R(\beta)+U(\alpha)-\eta T(\beta)>0\\ \mathrm{tr}(S(\beta))=1.\end{array}\right. \tag{12}$$

Let us define

$$V=R(\beta^*)+U(\alpha^*) \tag{13}$$

where α^*,β^* are optimal values of α,β in (12), and let $c_1,\ldots,c_r$ be the eigenvectors of the non-positive eigenvalues of V, i.e.

$$\left\{\begin{array}{l}c_i'c_i=1\\ Vc_i=\lambda_ic_i\text{ for some }\lambda_i\in\mathbb{R},\ \lambda_i\leq 0.\end{array}\right. \tag{14}$$

Theorem 2 *The set $\mathcal{A}$ is stable if and only if there exists m such that $\eta^* > 0$. Moreover, $\mathcal{A}$ is unstable if and only if there exist m and $(u,y) \in \mathbb{R}_0^q \times \mathbb{R}_0^n$ such that $A(\xi(u))$ is unstable and*

$$u^{\{m+d\}} \otimes y \ \in \ \operatorname{span}\{c_1,\ldots,c_r\} \tag{15}$$

where $\xi : \mathbb{R}_0^q \to \mathcal{P}$ is the function

$$\xi(u) = \left(\sum_{i=1}^{q} u_i^2\right)^{-1} \operatorname{sq}(u). \tag{16}$$

Proof. Let us consider the stability statement, and let us observe that K in (11) is a full column rank (see e.g. [3]), which directly implies from (10) that

$$T(\beta) > 0 \iff S(\beta) > 0. \tag{17}$$

Therefore, the stability statement follows from (17) and Theorem 1. Indeed, observe that the constraint $\operatorname{tr}(S(\beta)) = 1$ is not restrictive since $S(\beta)$, $R(\beta)$, $U(\alpha$ and $T(\beta)$ are linear functions, and it is introduced in order to normalized the solution of (12).

Let us consider the instability statement. The sufficiency is obvious because, if $A(\xi(u))$ is unstable and $\xi(u) \in \mathcal{P}$, then $\mathcal{A}$ is unstable for definition. Hence, let us consider the necessity and let us assume that $\mathcal{A}$ is unstable. From the stability statement, we have $\eta^* \leq 0$. Observe that

$$\Delta\left(p^{\{m+d\}}, T(\beta)\right) = \left(\sum_{i=1}^{q} p_i^2\right)^{d} P(\operatorname{sq}(p),\beta) \tag{18}$$

and let us suppose for contradiction that, for all m, (15) does not hold.

Let us consider firstly the CT case. This supposition implies that $\operatorname{Re}(\lambda) < -0.5\eta^*$ for all $\lambda \in \operatorname{spc}(A(p))$ for all $p \in \mathcal{P}$. In fact, from (6)–(8), (18) and Lemma 3 in [3], the first two constraints in (12) imply

$$\begin{cases} P(p) > 0 \\ Q(p) - \eta P(p) > 0 \end{cases} \quad \forall p \in \mathcal{P}. \tag{19}$$

Consequently, there exists $\varepsilon > 0$ such that $A(p) + 0.5(\eta^* + \varepsilon)I$ is stable for all $p \in \mathcal{P}$. Let us replace $A(p)$ with $A(p) + 0.5(\eta^* + \varepsilon)I$ in our original problem. It follows that the new set $\mathcal{A}$ is stable, and the new solution of (12), which we refer to as $\eta^{\#}$, satisfies $\eta^{\#} = -\varepsilon$. But since $\varepsilon > 0$ this implies that (9) is not satisfied for any m, hence contradicting Theorem 1.

Let us consider now the DT case. The supposition that (15) does not hold for any m implies that $|\lambda| < \sqrt{1-\eta^*}$ for all $\lambda \in \operatorname{spc}(A(p))$ for all $p \in \mathcal{P}$. Starting from this fact, one can construct another problem leading to a contradiction and hence proving the statement. The construction of this problem is completely similar to the one in the CT case, and it is omitted for conciseness. □

Theorem 2 requires to solve (12), which is a generalized eigenvalue problem and hence a quasi-convex optimization problem. The pairs (u,y) in $\mathbb{R}_0^q \times \mathbb{R}_0^n$ satisfying (15) can be found with the technique in [19, 12] which amounts to finding the roots of a polynomial obtained via pivoting. The vectors $c_1,\ldots,c_r$ can be obtained once (12) has been solved, being eigenvectors of V.

In order to clarify the construction of (12), let us consider a simple situation with $n = 2$, $q = 2$, $m_A = 1$ and $m = 1$ in the CT case. Let us select $p^{\{m\}} = (p_1, p_2)'$ and $p^{\{m+d\}} = (p_1^2, p_1p_2, p_2^2)'$. A parametrization $S(\beta)$ for the set $\mathcal{S}$ in (4) and the matrix K in (11) are hence

$$S(\beta) = \begin{pmatrix} \beta_1 & \beta_2 & 0 & -\beta_7 \\ \star & \beta_3 & \beta_7 & 0 \\ \star & \star & \beta_4 & \beta_5 \\ \star & \star & \star & \beta_6 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, the matrix $T(\beta)$ is directly given by (10). Lastly, the matrices $R(\beta)$ and $U(\alpha)$ can be computed as described in [3, 12] and are omitted for conciseness.

Remark 1. The proposed condition provides a solution for the decidability problem of establishing whether $A(p)$ is either stable or unstable. Indeed, for any chosen degree m of the HPD-QLF, Theorem 2 allows one to establish either stability (if $\eta^* > 0$) or instability (if (15) holds). Moreover, the theorem guarantees that, for a finite m, one of these conditions is satisfied.

4 Examples

Here we present some illustrative examples. The computational time of the proposed condition for all these examples (and of [17] for the examples in Section 4.1) is in the range 1–3 seconds.

4.1 Linear Dependence

Consider in the DT case the example of Figure 3 in [17], where $n = 3$, $q = 3$ and $m_A = 1$. With $m = 1$ Theorem 2 proves that $\mathcal{A}$ is unstable, in particular we obtain

$$\begin{array}{rcl} \xi(u) & = & (0.0821, 0.1425, 0.7754)' \\ \mathrm{spc}(A(\xi(u))) & = & \{-1.0002, 0.1863 \pm 0.5453i\}. \end{array}$$

Some details are: $\beta \in \mathbb{R}^{26}$; $\alpha \in \mathbb{R}^{297}$; u is found from (15) with the technique in [19, 12] by finding the roots of a cubic polynomial.

Still in the DT case, consider the example of Figure 5 in [17], where $n = 3$, $q = 4$ and $m_A = 1$. With $m = 0$ Theorem 2 proves that $\mathcal{A}$ is unstable, in particular

$$\begin{array}{rcl} \xi(u) & = & (0.3172, 0.6828, 0.0000, 0.0000)' \\ \mathrm{spc}(A(\xi(u))) & = & \{0.0910, -0.6655, -1.0560\}. \end{array}$$

Some details are: $\beta \in \mathbb{R}^{5}$; $\alpha \in \mathbb{R}^{255}$; u is found from (15) with the technique in [19, 12] by finding the roots of a quartic polynomial.

4.2 Nonlinear Dependence

Here the aim is to show that the proposed approach works well also in the case where the portion of the simplex yielding an unstable system is extremely small, in particular a point. Consider in the CT case the problem of establishing whether $M(\theta)$ is stable for all $\theta \in \{\theta \in \mathbb{R}^2 : \ \theta_1 + \theta_2 \le 1, \ \theta_i \ge 0\}$, where

$$\begin{array}{rcl} M(\theta) & = & M_0 + M_i \sum_{i=1}^{2} (\theta_i - \theta_i^*)^2 \\ \mathrm{vec}(M_0) & = & (-1.1, 0, 0, -1.2, 1.2, 1.2, -4.6, -5.5, -5.5)' \\ \mathrm{vec}(M_1) & = & (-1.7, 0.3, 0, 0, -1.5, 0.5, 0.5, 0.2, -1.3)' \\ \mathrm{vec}(M_2) & = & (-1.5, 0, 0.3, 0, -1.8, 0.2, -0.6, 0, -1.4)' \\ \theta^* & = & (\sqrt{2}/5, \pi/9)'. \end{array}$$

It can be verified that $M(\theta)$ is stable for all admissible θ except $\theta = \theta^*$. This problem can be addressed with the proposed approach. Indeed, let us define

$$\begin{array}{c} A(p) = M_0 s(p)^2 + M_i \sum_{i=1}^{2} \left(p_i - \theta_i^* s(p)\right)^2 \\ p_1 = \theta_1, \ p_2 = \theta_2, \ p_3 = 1 - \theta_1 - \theta_2, \ s(p) = \sum_{i=1}^{3} p_i. \end{array}$$

We have $n = 3$, $q = 3$ and $m_A = 2$. Theorem 2 proves that $\mathcal{A}$ is unstable with $m = 0$, in particular we obtain

$$\begin{array}{rcl} \xi(u) & = & (0.2828, 0.3491, 0.3681)' \\ \mathrm{spc}(A(\xi(u))) & = & \{-1.1000, 0.0000, -4.3000\}. \end{array}$$

Some details are: $\beta \in \mathbb{R}^5$; $\alpha \in \mathbb{R}^{81}$; u is found from (15) with the technique in [19, 12] by finding the roots of a quadratic polynomial.

For completeness, we slightly modify $M(\theta)$ in order to consider a stable system, in particular we redefine M_0 as $M_0 - 10^{-3} I_3$. We find that $\eta^* > 0$, which implies from Theorem 2 that $\mathcal{A}$ is stable.

5 Conclusion

We have proposed a condition via HPD-QLFs for establishing stability and instability of linear systems with polynomial dependence on uncertainties constrained in the simplex. This condition is necessary and sufficient.

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