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Fixed-Order $\mathcal{H}_\infty$ Filtering for Discrete-Time Markovian Jump Linear Systems with Unobservable Jump Modes

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***Abstract*— In practical applications, it is often encountered that the jump modes of a Markovian jump linear system may not be fully accessible to the filter, and thus designing a filter which partially or totally independent of the jump modes becomes significant. In this paper, by virtue of a new stability and $\mathcal{H}_\infty$ performance characterization, a novel necessary and sufficient condition for the existence of mode-independent $\mathcal{H}_\infty$ filters is established in terms of a set of nonlinear matrix inequalities that possess special properties for computation. Then, two computational approaches are developed to solve the condition. One is based on the solution of a set of linear matrix inequalities (LMIs), and the other is based on the sequential LMI optimization with more computational effort but less conservatism. In addition, a specific property of the feasible solutions enables one to further improve the solvability of these two computational approaches.**

***Index Terms*—Fixed-order filter, $\mathcal{H}_\infty$ filtering, iterative calculation, linear matrix inequality (LMI), Markovian jump linear systems, mode-independence.**

I. INTRODUCTION

As one of the most important robust filtering approaches, $\mathcal{H}_\infty$ filtering has received tremendous attention over the past decade. Compared with the well-known Kalman filtering approach, $\mathcal{H}_\infty$ filtering is insensitive to the system date and does not require exact statistics of the noise signals, and thus making it specific for applications where the system parameters and the statistics of the noise signals are not exactly known. Some classical results are available in [1], [2], [3] and references therein.

On the other hand, various practical processes may experience abrupt changes in their structure and parameters, possibly caused by phenomena such as component failures or repairs, sudden environmental disturbances, changing subsystem interconnections. On account of these changes, the resulting system models of these processes may have a hybrid characteristic. An important category of such hybrid models is the Markovian jump linear system (MJLS) model, which contains a set of linear systems with transitions among the models determined by a Markov chain taking values in a finite set. This type of systems has gained considerable interest for many years, and a lot of important results on stability, linear quadratic regulation, and $\mathcal{H}_2/\mathcal{H}_\infty$ control have been obtained [4], [5], [6], [7], [8]. In particular, Kalman filtering and least minimum mean square filtering have been treated in [9] and [10], respectively, while some results on $\mathcal{H}_\infty$ filtering can be found in [11], [12], [13], [14] and references therein.

In most existing works, it is often assumed that the jump modes are completely accessible to the filter. In practice, however, this assumption may not be always true, and it is necessary to consider the more practical case that the jump modes are partially accessible or inaccessible. Moreover, the deterministic (mode-independent) filter, compared with the stochastic (mode-dependent) filter, would have the advantages of simplicity, lower cost, and higher reliability, provided that it can generate a satisfactory performance. Hence, the design of mode-independent filters has received an increasing deal of attention. A natural approach to this problem is to use a mode-independent Lyapunov function [15], but this will introduce significant restriction on performance condition which in turn makes the corresponding design condition very conservative. Recently, a structural Lyapunov matrix approach has been proposed to design the mode-independent filter in [16], and the conservatism has been reduced. However, this approach are not applicable to MJLSs with the so-called "terminal mode", and thus some improvements via the slack matrix technique have been developed in [17]. Despite the effectiveness of these approaches, there are still some critical issues that has not be treated well. On one hand, to make the design condition numerically tractable, various constraints have been imposed on the Lyapunov matrices. On the other hand, only the full-order filter case has been considered, while the result of the reduced-order case is still unavailable. In addition, structural gain constraints, which are common in practical applications, for example, decentralized control and signal processing, cannot be incorporated in a straightforward manner via these approaches. All of these form the motivation of the present study.

In this paper, we investigate the fixed-order $\mathcal{H}_\infty$ filtering problem for discrete-time MJLSs with unobservable jump modes. By virtue of a new stability and $\mathcal{H}_\infty$ performance characterization, a novel necessary and sufficient condition for the existence of mode-independent $\mathcal{H}_\infty$ filters is established in terms of a set of nonlinear matrix inequalities. In the new condition, the filter gain is parametrized by a mode-independent free positive definite matrix, and no constraints are imposed on the multiple Lyapunov matrices. Then, two computational approaches are developed from some properties of the design condition. One is based on the solution of a set of linear matrix inequalities, and the other is based on the sequential LMI optimization with more computational effort but less conservatism. In addition, a

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specific characteristic of the feasible solutions enables one to further improve the solvability of these two computational approaches.

Notation: The superscript "T" represents the transpose of a matrix. Let $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the *n*-dimensional real Euclidean space equipped with the norm $\|x\| = \sqrt{x^T x}$ and the $m \times n$ real matrix space equipped with the spectral norm $\|A\| = \sup\{\|Ax\| \mid x \in \mathbb{R}^n, \|x\| = 1\}$, respectively. For symmetric matrices X, $Y \in \mathbb{R}^{n\times n}$, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). For a matrix $C \in \mathbb{R}^{m\times n}$, $C^\perp$ denotes the orthogonal complement of C. For a matrix $A \in \mathbb{R}^{n\times n}$, $\mathrm{Herm}(A) = A + A^T$. $\mathbb{E}[\cdot]$ stands for the mathematical expectation with respect to the underlying probability measure. l_2^n refers to the space of *n*-dimensional mean square summable vector sequences $f \triangleq (f(0), f(1), \ldots)$, $f(k) \in \mathbb{R}^n$ equipped with the norm $\|f\|_2 = \sqrt{\sum_{k=0}^{\infty} \mathbb{E}\left[\|f(k)\|^2\right]}$. The symbol $\#$ is used to denote a matrix which can be inferred by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

II. PRELIMINARIES AND PROBLEM FORMULATION

On a complete probability space, consider a class of discrete-time MJLSs of the following form:

$$\begin{cases} x(k+1) = A_{\theta(k)}x(k) + B_{w\theta(k)}w(k), \\ z(k) = C_{\theta(k)}x(k) + D_{w\theta(k)}w(k), \\ y(k) = C_{y\theta(k)}x(k) + D_{yw\theta(k)}w(k), \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $w(k) \in \mathbb{R}^{n_w}$, $y(k) \in \mathbb{R}^{n_y}$, and $z(k) \in \mathbb{R}^{n_z}$ are the system state, the disturbance input belonging to $l_2^{n_w}$, the measured output, and the signals to be estimated, respectively; the parameter $\theta(k)$ represents a discrete-time homogeneous Markov chain taking values in a finite set $\mathcal{S} = \{1, 2, \ldots, s\}$ with transition probability matrix $\mathbb{P} = [p_{ij}]$, that is, $\Pr\{\theta(k+1) = j \mid \theta(k) = i\} = p_{ij} \geq 0$, $\sum_{j=1}^{s} p_{ij} = 1$, $i, j \in \mathcal{S}$.

The filter under consideration is of the form

$$\begin{cases} \eta(k+1) = F_A\eta(k) + F_B y(k), \\ \hat{z}(k) = F_C\eta(k) + F_D y(k), \end{cases} \tag{2}$$

where $\eta(k) \in \mathbb{R}^{n_f}$, and $\hat{z}(k) \in \mathbb{R}^{n_{\hat{z}}}$ is the estimator of $z(k)$, and F_A, F_B, F_C, F_D are the filter parameter matrices to be designed. By defining the filtering error as $e(k) = z(k) - \hat{z}(k)$, and connecting filter (2) to the original system, one obtains the following error dynamic system:

$$\begin{cases} x_f(k+1) = \left(\bar{A}_{\theta(k)} + \bar{B}_{\theta(k)}F\bar{C}_{y\theta(k)},\right) x_f(k) \\ \qquad + \left(\bar{B}_{w\theta(k)} + \bar{B}_{\theta(k)}F\bar{D}_{yw\theta(k)}\right) w(k), \\ e(k) = \left(\bar{C}_{\theta(k)} + \bar{D}_{\theta(k)}F\bar{C}_{y\theta(k)}\right) x_f(t) \\ \qquad + \left(\bar{D}_{w\theta(k)} + \bar{D}_{\theta(k)}F\bar{D}_{yw\theta(k)}\right) w(k), \end{cases} \tag{3}$$

where $x_f(k) = \begin{bmatrix} x^T(k) & \eta^T(k) \end{bmatrix}^T$, and

$$\begin{aligned} \bar{A}_{\theta(k)} &= \begin{bmatrix} A_{\theta(k)} & 0 \\ 0 & 0 \end{bmatrix}, & \bar{B}_{\theta(k)} &= \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \\ \bar{C}_{\theta(k)} &= \begin{bmatrix} C_{\theta(k)} & 0 \end{bmatrix}, & \bar{D}_{\theta(k)} &= \begin{bmatrix} 0 & D_{\theta(k)} \end{bmatrix}, \\ \bar{B}_{w\theta(k)} &= \begin{bmatrix} B_{w\theta(k)} \\ 0 \end{bmatrix}, & \bar{D}_{w\theta(k)} &= D_{w\theta(k)}, \\ \bar{C}_{y\theta(k)} &= \begin{bmatrix} 0 & I \\ C_{y\theta(k)} & 0 \end{bmatrix}, & \bar{D}_{yw\theta(k)} &= \begin{bmatrix} 0 \\ D_{yw\theta(k)} \end{bmatrix}, \\ F &= \begin{bmatrix} F_A & F_B \\ F_C & F_D \end{bmatrix}. \end{aligned}$$

For the filtering system in (3), we have the following definitions.

Definition 1: The filtering system in (3) is said to be stochastically stable if, when $w(k) \equiv 0$, the following inequality holds for any $x_f(0) = x_{f0}$ and $\theta(0) = \theta_0$,

$$\lim_{N\to\infty} \left\{ \sum_{k=0}^{N} \mathbb{E}\left[\|x(k)\|^2 \mid x_0, \theta_0\right] \right\} < \infty.$$

Definition 2: Let $\mathcal{T}_{ew}$ denote the operator from w to e; the $\mathcal{H}_\infty$ norm of the operator $\mathcal{T}_{ew}$ is defined as

$$\|\mathcal{T}_{ew}\|_\infty \triangleq \sup_{\theta(0)\in\mathcal{S}} \sup_{0\neq w\in l_2^{n_w}} \frac{\|e\|_2}{\|w\|_2}.$$

The purpose of the paper is to design a filter in (2) of order n_η (probably less than n) such that the filtering system in (3) is stochastically stable and $\|\mathcal{T}_{ew}\|_\infty < \gamma$, which is a prescribed specification.

Throughout the paper, we shall make the following assumption.

Assumption: The filtering system in (3) is weakly controllable[1].

The assumption ensures that the disturbance can affect the system state, and is essential for the necessity of the following bounded real lemma.

Lemma 1 ([8]): Assume that the filtering system is weakly controllable. It is stochastically stable with $\|\mathcal{T}_{ew}\|_\infty < \gamma$ if and only if there exist matrices $P_i > 0$ such that, for each $i \in \mathcal{S}$,

$$\Psi_i = \begin{bmatrix} A_{fi} & B_{fwi} \\ C_{fi} & D_{fwi} \end{bmatrix}^T \begin{bmatrix} \hat{P}_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{fi} & B_{fwi} \\ C_{fi} & D_{fwi} \end{bmatrix} - \begin{bmatrix} P_i & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0.$$

where $\hat{P}_i = \sum_{j=1}^{s} p_{ij}P_j$ and

$$\begin{aligned} A_{fi} &= \bar{A}_i + \bar{B}_i F \bar{C}_{yi}, & B_{fwi} &= \bar{B}_{wi} + \bar{B}_i F \bar{D}_{ywi}, \\ C_{fi} &= \bar{C}_i + \bar{D}_i F \bar{C}_{yi}, & D_{fwi} &= \bar{D}_{wi} + \bar{D}_i F \bar{D}_{ywi}. \end{aligned}$$

[1]The filtering system is said to be weakly controllable with respect to w if for every initial state/mode, (x_{f0}, θ_0), and any final state/mode, (x_{fd}, θ_d), there exists a finite time t_c and an input w such that $\Pr[x_f(t_c) = x_{fd}, \theta(t_c) = \theta_d] > 0$. Please refer to [8] and reference therein for more details

III. NEW STABILITY AND $\mathcal{H}_\infty$ PERFORMANCE CHARACTERIZATION

Theorem 1: The following statements are equivalent:

1) The filtering system in (3) is stochastically stable with $\|\mathcal{T}_{ew}\|_\infty < \gamma$.
2) There exist $P_i = P_i^T$, $P_{2i} = P_{2i}^T$, P_{1i}, $Q > 0$, and a scalar $\alpha > 0$ such that, for each $i \in \mathcal{S}$,

$$\begin{aligned} \mathcal{P}_i &> 0 && (4) \\ \Omega_i &= \mathcal{A}_i^T \hat{\mathcal{P}}_i \mathcal{A}_i - \mathcal{E}_i^T \mathcal{P}_i \mathcal{E}_i + \alpha \,\mathrm{Herm}\,(\mathcal{Q}_i \mathcal{L}_i) < 0. && (5) \end{aligned}$$

where

$$\mathcal{A}_i = \begin{bmatrix} \bar{A}_i & \bar{B}_i & \bar{B}_{wi} \\ F\bar{C}_{yi} & -I & F\bar{D}_{ywi} \\ \bar{C}_i & \bar{D}_i & \bar{D}_{wi} \end{bmatrix}, \quad \mathcal{E}_i = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma I \end{bmatrix},$$

$$\mathcal{P}_i = \begin{bmatrix} P_i & P_{1i}^T & 0 \\ P_{1i} & P_{2i} & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \hat{\mathcal{P}}_i = \sum_{j=1}^{s} p_{ij} \mathcal{P}_j,$$

$$\mathcal{L}_i = \begin{bmatrix} \bar{A}_i & \bar{B}_i & 0 \\ F\bar{C}_{yi} & -I & F\bar{D}_{ywi} \end{bmatrix},$$

$$\mathcal{Q}_i = \begin{bmatrix} 0 & -\bar{C}_{yi}^T F^T Q \\ 0 & Q \\ 0 & -\bar{D}_{ywi}^T F^T Q \end{bmatrix}.$$

3) There exist $P_i = P_i^T$, $P_{2i} = P_{2i}^T$, P_{1i}, $Q > 0$, and a scalar $\alpha > 0$ such that, for each $i \in \mathcal{S}$,

$$\begin{aligned} \mathcal{P}_i &> 0, && (6) \\ \left(\mathcal{Z}_i^\perp\right)^T \begin{bmatrix} \Omega_{1i} & \# \\ \mathcal{G}\mathcal{A}_i & \hat{\mathcal{P}}_i - 2\mathcal{G} \end{bmatrix} \mathcal{Z}_i^\perp &< 0. && (7) \end{aligned}$$

where

$$\begin{aligned} \Omega_{1i} &= \alpha \,\mathrm{Herm}\,(\mathcal{Q}_i \mathcal{L}_i) - \mathcal{E}_i^T \mathcal{P}_i \mathcal{E}_i, \\ \mathcal{G} &= \mathrm{diag}\,(0, Q, 0), \\ \mathcal{Z}_i &= \begin{bmatrix} \bar{A}_i & \bar{B}_i & \bar{B}_{wi} & -I & 0 & 0 \\ \bar{C}_i & \bar{D}_i & \bar{D}_{wi} & 0 & 0 & -I \end{bmatrix}. \end{aligned}$$

Proof: 3)⇒2) It follows from (7) and the Matrix Elimination Lemma [18] that there exist matrices Δ_i such that

$$\begin{bmatrix} \Omega_{1i} & \# \\ \mathcal{G}\mathcal{A}_i & \hat{\mathcal{P}}_i - 2\mathcal{G} \end{bmatrix} + \mathrm{Herm}\left(\mathcal{Z}_i^T \Delta_i\right) < 0. \qquad (8)$$

Pre- and post-multiplying (8) by $\begin{bmatrix} I & \mathcal{A}_i^T \end{bmatrix}$ and its transpose, and noticing that $\begin{bmatrix} I & \mathcal{A}_i^T \end{bmatrix} \mathcal{Z}_i^T = 0$, one has that (5) holds.

2)⇒1) Define a nonsingular transformation matrix as follows:

$$T_i = \begin{bmatrix} I & 0 & 0 \\ F\bar{C}_{yi} & F\bar{D}_{ywi} & I \\ 0 & I & 0 \end{bmatrix}.$$

Pre- and post-multiplying (5) by T_i^T and its transpose yields that

$$\begin{aligned} & T_i^T \Omega_i T_i \\ = & \begin{bmatrix} \begin{pmatrix} A_{fi}^T \hat{P}_i A_{fi} - P_i \\ + C_{fi}^T C_{fi} \end{pmatrix} & \# & \# \\ \begin{pmatrix} B_{fwi}^T \hat{P}_i A_{fi} \\ + D_{fwi}^T C_{fi} \end{pmatrix} & \begin{pmatrix} -\gamma^2 I \\ + B_{fwi}^T \hat{P}_i B_{fwi} \\ + D_{fwi}^T D_{fwi} \end{pmatrix} & \# \\ \begin{pmatrix} \bar{B}_i^T \hat{P}_i A_{fi} \\ - \hat{P}_{1i} A_{fi} \\ + \bar{D}_i^T C_{fi} \end{pmatrix} & \begin{pmatrix} \bar{B}_i^T \hat{P}_i B_{fwi} \\ - \hat{P}_{1i} B_{fwi} \\ + \bar{D}_i^T D_{fwi} \end{pmatrix} & \begin{pmatrix} \bar{B}_i^T \hat{P}_i \bar{B}_i \\ - \bar{B}_i^T \hat{P}_{1i}^T - \hat{P}_{1i} \bar{B}_i \\ + \hat{P}_{2i} + \bar{D}_i^T \bar{D}_i \\ - 2\alpha Q \end{pmatrix} \end{bmatrix} \\ < & \ 0, \end{aligned} \qquad (9)$$

where $\hat{P}_i$ is defined in Lemma 1, and $\hat{P}_{1i}$ and $\hat{P}_{2i}$ are defined similarly. The 2nd leading principal submatrix of (9) implies that $\Psi_i < 0$, for each $i \in \mathcal{S}$. Therefore, according to Lemma 1, the filtering system in (3) is stochastically stable with $\|\mathcal{T}_{ew}\|_\infty < \gamma$.

1)⇒3) If 1) holds, then, according to Lemma 1, there exist matrices $P_i > 0$ such that $\Psi_i < 0$. Now set $P_{1i} = 0$, $Q = P_{21} = P_{22} = \cdots = P_{2s} > 0$ to be an arbitrary matrix, and $\alpha > 0$ to be a sufficiently large scalar such that

$$-\mathcal{F}_i^T \Psi_i^{-1} \mathcal{F}_i + \bar{B}_i^T \hat{P}_i \bar{B}_i + \bar{D}_i^T \bar{D}_i - (2\alpha - 1) Q < 0, \quad (10)$$

where $\mathcal{F}_i = \begin{bmatrix} A_{fi}^T \hat{P}_i \bar{B}_i + C_{fi}^T \bar{D}_i \\ B_{fwi}^T \hat{P}_i \bar{B}_i + D_{fwi}^T \bar{D}_i \end{bmatrix}$. Direct manipulations together with (9), (10), and Schur complement equivalence yield that

$$\begin{aligned} & \Omega_i \\ = & \ T_i^{-T} \begin{bmatrix} \Psi_i & \mathcal{F}_i \\ \mathcal{F}_i^T & \bar{B}_i^T \hat{P}_i \bar{B}_i + \bar{D}_i^T \bar{D}_i - (2\alpha - 1) Q \end{bmatrix} T_i^{-1} \\ < & \ 0. \end{aligned}$$

Define matrices Δ_i, $i \in \mathcal{S}$, as follows:

$$\Delta_i = \begin{bmatrix} 0 & 0 & 0 & \hat{P}_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

where the partition is compatible with $\mathcal{Z}_i$. Then, algebraic manipulations together with Schur complement equivalence give that

$$\begin{aligned} & \begin{bmatrix} \Omega_{1i} & \# \\ \mathcal{G}\mathcal{A}_i & \hat{\mathcal{P}}_i - 2\mathcal{G} \end{bmatrix} + \mathrm{Herm}\left(\mathcal{Z}_i^T \Delta_i\right) \\ = & \begin{bmatrix} \Omega_{1i} & \# \\ \hat{\mathcal{P}}_i \mathcal{A}_i & -\hat{\mathcal{P}}_i \end{bmatrix} \\ < & \ 0, \end{aligned}$$

which, by the Matrix Elimination Lemma [18], implies that 3) holds. This completes the proof.

Remark 1: The significance of the conditions in Theorem 1 lies in two aspects. On one hand, the separation of the multiple Lyapunov matrices P_i and the single filter matrix F avoids imposing any constraint on P_i when F is parametrized. On the other hand, an arbitrary mode-independent matrix Q has been introduced without loss of generality to parametrize F, and the arbitrariness of Q enables one to add other specifications such as structural gain into design procedure in a straightforward manner.

Remark 2: Although conditions 2) and 3) are equivalent, whereas, 3) may be more desirable from a computational point of view. This is because the multipliers Δ_i, also referred to as slack matrix variables in some cases, have been introduced in 3), and they are expected to make the computation to be presented later more efficient and less conservative.

IV. DESIGN CONDITION AND COMPUTATION

A. Design Condition

Theorem 2: There exists a filter in (2) such that the filtering system in (3) is stochastically stable with $\|\mathcal{T}_{ew}\|_\infty < \gamma$ if and only if there exist $P_i = P_i^T$, $P_{2i} = P_{2i}^T$, P_{1i}, M_i, N_i, $i \in \mathcal{S}$, $Q > 0$, L, and a scalar $\alpha > 0$ such that, for each $i \in \mathcal{S}$,

$$\mathcal{P}_i > 0, \quad (11)$$

$$\left(\mathcal{Z}_i^\perp\right)^T \begin{bmatrix} \Xi_{1i}(\alpha, M_i, N_i) & \# \\ \Xi_{2i} & \hat{\mathcal{P}}_i - 2\mathcal{G} \end{bmatrix} \mathcal{Z}_i^\perp < 0, \quad (12)$$

where

$$\begin{aligned} \Xi_{1i}(\alpha, M_i, N_i) &= \Pi_i(\alpha, M_i, N_i) - \mathcal{E}_i^T \mathcal{P}_i \mathcal{E}_i \\ \Pi_i(\alpha, M_i, N_i) &= 2\alpha\Upsilon_i - 2\alpha \operatorname{Herm}\left(\mathcal{C}_{yi}^T L^T \mathcal{M}_i\right) \\ &\quad + 2\alpha \mathcal{M}_i^T Q \mathcal{M}_i, \\ \Xi_{2i} &= \begin{bmatrix} 0 & 0 & 0 \\ L\bar{C}_{yi} & -Q & L\bar{D}_{ywi} \\ 0 & 0 & 0 \end{bmatrix}, \\ \Upsilon_i &= \begin{bmatrix} 0 & \bar{C}_{yi}^T L^T & 0 \\ L\bar{C}_{yi} & -Q & L\bar{D}_{ywi} \\ 0 & \bar{D}_{ywi}^T L^T & 0 \end{bmatrix}, \\ \mathcal{C}_{yi} &= \begin{bmatrix} \bar{C}_{yi} & 0 & \bar{D}_{ywi} \end{bmatrix}, \\ \mathcal{M}_i &= \begin{bmatrix} M_i & 0 & N_i \end{bmatrix}. \end{aligned}$$

Under the condition, a mode-independent filter matrix can be obtained as

$$F = Q^{-1} L. \quad (13)$$

Proof: It suffices to prove (7) and (12) are equivalent.

(Sufficiency) $Q > 0$ implies that (13) is meaningful. By expanding (7) and noting that, for any M_i and N_i, $-\mathcal{C}_{yi}^T F^T Q F \mathcal{C}_{yi} \le -\operatorname{Herm}\left(\mathcal{C}_{yi}^T L^T \mathcal{M}_i\right) + \mathcal{M}_i^T Q \mathcal{M}_i$, one obtains that (7) holds if (12) holds, where the parametrization $L = QF$ is used.

(Necessity) Assume that (7) holds. Then, by choosing $M_i = F\bar{C}_{yi}$ and $N_i = F\bar{D}_{ywi}$, one has that

$$\begin{aligned} & -\mathcal{C}_{yi}^T F^T Q F \mathcal{C}_{yi} \\ = & -\mathcal{C}_{yi}^T F^T Q F \mathcal{C}_{yi} + \left(\mathcal{M}_i - F\mathcal{C}_{yi}\right)^T Q \left(\mathcal{M}_i - F\mathcal{C}_{yi}\right) \\ = & -\operatorname{Herm}\left(\mathcal{C}_{yi}^T F^T Q \mathcal{M}_i\right) + \mathcal{M}_i^T Q \mathcal{M}_i. \end{aligned}$$

Substituting this into (7), and letting $L = QF$, one obtains that (12) holds. This completes the proof.

B. LMI-Type Computation

When α, M_i, and N_i are fixed, (12) becomes linear with respect to the decision variables for each mode. A natural approach is to fix α, M_i, and N_i, and try to solve (12). Although randomly selected α, M_i, and N_i may generate a feasible solution, the solvability, that is, the success rate in some sense, might not be high. To increase the solvability, some conditions that α, M_i, and N_i must be satisfied are given in the following proposition.

Proposition 1: If there exists a desired filter, then (11)–(12) hold for some $\alpha > 0$, M_i and N_i, where α is a sufficiently large scalar, and M_i, N_i are matrices such that the following system is stochastically stable with $\|\mathcal{G}_{ew}\|_\infty < \gamma$,

$$\mathcal{G}: \begin{cases} x_f(k+1) = \left(\bar{A}_{\theta(k)} + \bar{B}_{\theta(k)} M_{\theta(k)}\right) x_f(k) \\ \qquad + \left(\bar{B}_{w\theta(k)} + \bar{B}_{\theta(k)} N_{\theta(k)}\right) w(k), \\ e(k) = \left(\bar{C}_{\theta(k)} + \bar{D}_{\theta(k)} M_{\theta(k)}\right) x_f(t) \\ \qquad + \left(\bar{D}_{w\theta(k)} + \bar{D}_{\theta(k)} N_{\theta(k)}\right) w(k). \end{cases} \quad (14)$$

This is an immediate result from the proof of Theorems 1 and 2. On the ground of this proposition, a 2-step computational approach can be constructed.

- *LMI-Type Computation*
  1) Set $\alpha > 0$ to be a large value. Find M_i and N_i such that the $\mathcal{G}$ system in (14) is stochastically stable with $\|\mathcal{G}_{ew}\|_\infty < \gamma$, which can be done by some existing approaches.
  2) Substitute α, M_i, and N_i obtained in Step 1 to (11)–(12), and solve via existing algorithms such as the interior-point method.

If the LMI-type computation fails to find a feasible solution, one cannot conclude the nonexistence of a desired filter. For this case, one may either choose another set of α, M_i, and N_i, and restart the computation, or resort to the sequential LMI optimization presented in the sequel.

C. Sequential LMI Optimization

First, a monotonic property of (12) is provided in the following proposition.

Proposition 2: When other variables, that is, $P_i = P_i^T$, $P_{2i} = P_{2i}^T$, P_{1i}, $i \in \mathcal{S}$, $Q > 0$, and L, are fixed, the following relationship holds for any M_i, N_i, $i \in \mathcal{S}$, and $\alpha_M > \alpha_m > 0$,

$$\begin{aligned} & \Pi_i\left(\alpha_M, Q^{-1}L\bar{C}_{yi}, Q^{-1}L\bar{D}_{ywi}\right) \\ \le & \ \Pi_i\left(\alpha_m, Q^{-1}L\bar{C}_{yi}, Q^{-1}L\bar{D}_{ywi}\right) \\ \le & \ \Pi_i\left(\alpha_m, M_i, N_i\right). \end{aligned}$$

Proof: The second "$\le$" follows immediately from the proof of Theorem 2. As for the first "$\le$", direct algebraic operations give that

$$\begin{aligned} & \Pi_i\left(\alpha_M, Q^{-1}L\bar{C}_{yi}, Q^{-1}L\bar{D}_{ywi}\right) \\ & -\Pi_i\left(\alpha_m, Q^{-1}L\bar{C}_{yi}, Q^{-1}L\bar{D}_{ywi}\right) \\ = \ & 2\left(\alpha_m-\alpha_M\right)\begin{bmatrix} -\bar{C}_{yi}^T L^T \\ Q \\ -\bar{D}_{ywi}^T L^T \end{bmatrix} Q^{-1} \begin{bmatrix} -\bar{C}_{yi}^T L^T \\ Q \\ -\bar{D}_{ywi}^T L^T \end{bmatrix}^T \\ \leq \ & 0. \end{aligned}$$

From the proposition, it can be revealed that the scalar ϵ satisfying $\Pi_i(\alpha, M_i, N_i) < \epsilon I$ achieves its *global minimum* only if $\alpha \to +\infty$, $M_i = Q^{-1}L\bar{C}_{yi} = F\bar{C}_{yi}$, and $N_i = Q^{-1}L\bar{D}_{ywi} = F\bar{D}_{ywi}$. In view of this and aforementioned analysis, the following iterative algorithm is constructed to solve the condition of Theorem 2.

- *Sequential LMI Optimization*
  1) Set $\nu = 1$. Select $M_i^{(\nu)}$ and $N_i^{(\nu)}$ such that the system $\mathcal{G}$ in (14) is stochastically stable with $\|\mathcal{G}_{ew}\|_\infty < \gamma$. Choose appropriate $\alpha^{(\nu)} > 0$ and $\Delta\alpha > 0$. Set $\epsilon_*^{(\nu)} > 0$ to be a large number and $c > 0$ to be an arbitrary number.
  2) For fixed $\alpha^{(\nu)}$, $M_i^{(\nu)}$, and $N_i^{(\nu)}$, solve the following LMI optimization problem:
     Minimize ϵ subject to, $Q > 0$, (11) and

$$\begin{aligned} \text{left of (12)} &< \epsilon I, && (15) \\ \epsilon &\geq -c && (16) \end{aligned}$$

     Denote $\epsilon_*^{(\nu+1)}$, $Q^{(\nu)}$, and $L^{(\nu)}$as the optimal value of ϵ, Q, and L, respectively.
  3) If $\epsilon_*^{(\nu+1)} < 0$, then a desired filter matrix can be obtained as (13), that is, $F = \left(Q^{(\nu)}\right)^{-1} L^{(\nu)}$. **STOP**.
     Else if $\left|\epsilon_*^{(\nu+1)} - \epsilon_*^{(\nu)}\right| \leq \delta$, where δ is a prescribed tolerance, then go to Step 4, else update

$$\begin{aligned} M_i^{(\nu+1)} &= \left(Q^{(\nu)}\right)^{-1} L^{(\nu)}\bar{C}_{yi}, \\ N_i^{(\nu+1)} &= \left(Q^{(\nu)}\right)^{-1} L^{(\nu)}\bar{D}_{ywi}, \\ \alpha^{(\nu+1)} &= \alpha^{(\nu)} + \Delta\alpha, \end{aligned}$$

     and set $\nu = \nu + 1$, then go to 2).
  4) There may not exist a solution. **STOP** (or choose other $\alpha^{(1)}$, $\Delta\alpha$, and initial values $M_i^{(1)}$, $N_i^{(1)}$, then run the algorithm again).

Remark 3: It can be seen easily from Proposition 2 and (16) that the sequence $\epsilon_*^{(\nu)}$ is monotonic decreasing with respect to ν, that is, $\epsilon_*^{(\nu)} \leq \epsilon_*^{(\nu-1)}$, and bounded from below by $-c$. Therefore, the convergence of the iteration is guaranteed.

Remark 4: It is noted that the global optimality of the iteration is affected by the initial values $M_i^{(1)}$, $N_i^{(1)}$ and the tuning parameter $\alpha^{(\nu)}$, and is thus generally not guaranteed. Further optimization will be discussed in the next section.

V. DESIRABLE PARAMETERS FOR COMPUTATION

As mentioned previously, the initial values $M_i^{(1)}$, $N_i^{(1)}$, and the tuning parameter $\alpha^{(\nu)}$ may affect the global optimality of the iteration. To see this in a detailed way, let us consider

$$\begin{aligned} & -\alpha\mathcal{C}_{yi}^T F^T Q F \mathcal{C}_{yi} \\ \leq & -\alpha\,\mathrm{Herm}\left(\mathcal{C}_{yi}^T F^T Q^T \mathcal{M}_i\right) + \alpha\mathcal{M}_i^T S \mathcal{M}_i \\ = & -\alpha\mathcal{C}_{yi}^T F^T Q F \mathcal{C}_{yi} + \alpha\left(\mathcal{M}_i - F\mathcal{C}_{yi}\right)^T Q \left(\mathcal{M}_i - F\mathcal{C}_{yi}\right). \end{aligned}$$

It follows from this inequality that if there exists a desired F^*, then (11)–(12) will also be feasible, provided that $\left\|\alpha\left(\mathcal{M}_i - F^*\mathcal{C}_{yi}\right)^T Q \left(\mathcal{M}_i - F^*\mathcal{C}_{yi}\right)\right\|$ is sufficiently small. The converse is also true. In view of this, it is natural to improve the solvability of the iterative calculation by reducing $\left\|\alpha\left(\mathcal{M}_i - F^*\mathcal{C}_{yi}\right)^T Q \left(\mathcal{M}_i - F^*\mathcal{C}_{yi}\right)\right\|$, which can be achieved through adjusting the two parameters α and $\mathcal{M}_i$, namely, making α and $\|\mathcal{M}_i - F^*\mathcal{C}_{yi}\|$ sufficiently small. From Proposition 2, however, α should be large in order to achieve global optimality of the condition in Theorem 2. Hence, the only way is to reduce $\|\mathcal{M}_i - F^*\mathcal{C}_{yi}\|$ by choosing appropriate $\mathcal{M}_i$. Since

$$\begin{aligned} \mathcal{M}_i - K^*\mathcal{C}_{yi} &= \left(\mathbf{M}_i - K^*\mathbf{C}_{yi}\right)\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \\ \mathbf{M}_i &= \begin{bmatrix} M_i & N_i \end{bmatrix}, \\ \mathbf{C}_{yi} &= \begin{bmatrix} \bar{C}_{yi} & \bar{D}_{ywi} \end{bmatrix}, \end{aligned}$$

it suffices to reduce $\|\mathbf{M}_i - K^*\mathbf{C}_{yi}\|$. To this end, the following theorem, which plays a central role in selecting $\mathbf{M}_i$, is provided.

Theorem 3: For some matrices $\mathbf{M}_i$ and a scalar $\gamma > 0$, $i \in \mathcal{S}$, the following two statements are equivalent

1) There exists a desired filter matrix F^* satisfying $\|\mathbf{M} - F^*\mathbf{C}_y\| \leq \mu_1$, where $\mu_1 > 0$ is a sufficiently small scalar.
2) $(\mathbf{M}_i)_{i\in\mathcal{S}} \in \mathbb{S}_\infty$, and $\left\|\mathbf{M}\mathbf{C}_y^\perp\right\| \leq \mu_2$, where $\mu_2 > 0$ is a sufficiently small scalar, and

$$\mathbb{S}_\infty = \Big\{ (\mathbf{M}_i)_{i\in\mathcal{S}} \,|\text{the system } \mathcal{G} \text{ in (14) is stochastically stable with } \|\mathcal{G}_{ew}\|_\infty < \gamma \Big\},$$

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 & \cdots & \mathbf{M}_r \end{bmatrix}, \\ \mathbf{C}_y &= \begin{bmatrix} \mathbf{C}_{y1} & \mathbf{C}_{y2} & \cdots & \mathbf{C}_{yr} \end{bmatrix}. \end{aligned}$$

Proof: 1)⇒2) It follows from 1) that $\mathbf{M}_i = F^*\mathbf{C}_{yi} + \Sigma_i$, where Σ_i is a sufficiently small perturbation, and thus the system in (14) is still stochastically stable with $\|\mathcal{G}_{ew}\|_\infty < \gamma$. In addition, $\left\|\mathbf{M}\mathbf{C}_y^\perp\right\| = \left\|\left(\mathbf{M} - K^*\mathbf{C}_y\right)\mathbf{C}_y^\perp\right\| \leq \mu_1\left\|\mathbf{C}_y^\perp\right\| \triangleq \mu_2$, which is sufficiently small.

2)⇒1) It is noted that if $\mathrm{rank}\left(\mathbf{C}_y\right) = n_1 < n_y + n_f$, then $\mathbf{C}_y$ can be QR-factorized as

$$\mathbf{C}_y = \mathbf{U}\begin{bmatrix} \mathbf{C}_1 \\ 0 \end{bmatrix},$$

where $\mathbf{U} \in \mathbb{R}^{(n_y+n_f)\times(n_y+n_f)}$ is an orthogonal matrix, $\mathbf{C}_1 \in \mathbb{R}^{n_1\times r(n+n_f+n_w)}$ is a matrix with full row rank, and

$$\mathbf{C}_1^{\perp} = \mathbf{C}_y^{\perp}. \tag{17}$$

Now define F^* as

$$\begin{cases} \mathbf{M}\mathbf{C}_y^T\left(\mathbf{C}_y\mathbf{C}_y^T\right)^{-1}, & \text{if } \operatorname{rank}(\mathbf{C}_y) = n_y + n_f, \\ \left[\mathbf{M}\mathbf{C}_1^T\left(\mathbf{C}_1\mathbf{C}_1^T\right)^{-1} \quad 0\right]\mathbf{U}^T, & \text{if } \operatorname{rank}(\mathbf{C}_y) < n_y + n_f, \end{cases}$$

which implies that

$$\begin{cases} \mathbf{M}\mathbf{C}_y^T - K^*\mathbf{C}_y\mathbf{C}_y^T = \mathbf{0}, & \text{if } \operatorname{rank}(\mathbf{C}_y) = n_y + n_f, \\ \mathbf{M}\mathbf{C}_1^T - K^*\mathbf{C}_y\mathbf{C}_1^T = \mathbf{0}, & \text{if } \operatorname{rank}(\mathbf{C}_y) < n_y + n_f, \end{cases}$$

With this and (17), one obtains that

$$\begin{aligned} (\mathbf{M} - F^*\mathbf{C}_y) \times \left[\begin{array}{cc} \mathbf{C}_y^T & \mathbf{C}_y^{\perp} \end{array}\right] &= \left[\begin{array}{cc} 0 & \mathbf{M}\mathbf{C}_y^{\perp} \end{array}\right], \\ \text{if } \operatorname{rank}(\mathbf{C}_y) &= n_y + n_f, \\ (\mathbf{M} - F^*\mathbf{C}_y) \times \left[\begin{array}{cc} \mathbf{C}_1^T & \mathbf{C}_1^{\perp} \end{array}\right] &= \left[\begin{array}{cc} 0 & \mathbf{M}\mathbf{C}_y^{\perp} \end{array}\right], \\ \text{if } \operatorname{rank}(\mathbf{C}_y) &< n_y + n_f, \end{aligned}$$

which, by noting the invertibility of $\left[\begin{array}{cc} \mathbf{C}_y^T & \mathbf{C}_y^{\perp} \end{array}\right]$ and $\left[\begin{array}{cc} \mathbf{C}_1^T & \mathbf{C}_1^{\perp} \end{array}\right]$, implies that

$$\begin{aligned} & \|\mathbf{M} - F^*\mathbf{C}_y\| \\ \leq \;& \mu_2 \max\left\{\left\|\left[\begin{array}{cc} \mathbf{C}_y^T & \mathbf{C}_y^{\perp} \end{array}\right]^{-1}\right\|, \left\|\left[\begin{array}{cc} \mathbf{C}_1^T & \mathbf{C}_1^{\perp} \end{array}\right]^{-1}\right\|\right\} \\ \triangleq \;& \mu_1, \end{aligned}$$

which is sufficiently small. Similar to the derivation in 1)⇒2), one further obtains that F^* is a desired filter matrix.

From this theorem, one may conclude that desirable initial values should be in $\mathbb{S}_\infty$ and $\left\|\mathbf{M}\mathbf{C}_y^{\perp}\right\|$ should be small enough. For the extreme case that $(\mathbf{M}_i)_{i\in\mathcal{S}} \in \mathbb{S}_\infty$ and $\left\|\mathbf{M}\mathbf{C}_y^{\perp}\right\| = 0$, (11)–(12) must be feasible for a sufficiently large scalar $\alpha > 0$. Based on this, some optimization techniques such as D-K type iteration [19] or ellipsoidal approximation could be developed to find a desirable initial value. Details are omitted here due to page length consideration.

VI. CONCLUSIONS

On the ground of a new characterization for stochastic stability and $\mathcal{H}_\infty$ performance, a novel necessary and sufficient condition for the existence of fixed-order mode-independent filters has been established in terms of a set of nonlinear matrix inequalities, which possess some special properties for computation. Two approaches with different computational burdens and conservatism, that is, the LMI-type computation and the sequential LMI optimization, have been provided to solve the condition. In addition, a specific property of the feasible solutions enables one to further improve the solvability of these two computational approaches.

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