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Investigating the Load Flow Equations in Power Systems via LMI-based Techniques

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Abstract— **Solving the load flow equations is an important problem in power systems. This paper proposes an approach for addressing this problem via a convex optimization with LMI constraints. This approach ensures to find all solutions provided that the dimension of a linear space provided by the optimization is smaller than a known threshold. Then, this paper considers the characterization of the set of admissible power, in particular the computation of the largest admissible power, which amounts to solving a nonconvex optimization problem. It is shown that an upper bound of the sought power can be obtained via convex LMI optimization. Moreover, a necessary and sufficient condition for establishing whether the upper bound is tight or not is provided. Some numerical examples illustrate the proposed approach.**

I. Introduction

A fundamental problem in power systems consists of solving the load flow equations, which play a key role in energy management and transportation. These equations describe the relations between the powers injected into an interconnected power system formed by several buses and the phase angles of these buses. It is well-known that solving this problem is important for planning both long and short term operating plans, in order to provide at the same time efficient and economical solutions to electricity providers while delivering reliable service to electricity users.

Numerous and various methods have been derived and proposed in the last three decades in order to address the solution of the load flow equations, see for instance [1]–[7] and references therein. But unfortunately, despite all these efforts, the problem is still open.

Indeed, the load flow equations are a system of nonlinear (transcendental) equations, and as it is known there do not exist methods that guarantee to find all solutions for such a system, see for instance [8]. In fact, methods such as Newton-Raphson iterations allow one to find a subset of the set of sought solutions, but unfortunately cannot provide all sought solutions in general, and in any case do not enable one to conclude that all sought solutions have been found.

The load flow equations can be converted into a system of polynomial equations through suitable variable transformations and augmentations, but this strategy does not suffice to solve the problem. In fact, there exist analytical methods for solving a system of polynomial equations which are based on elimination theory such as polynomial resultants, see for instance [9], but such methods lead to prohibitive computational burdens except for systems where the number of variables and degree of the polynomials are very small, see for instance the numerical examples in Section IV. On the other hand, homotopy methods which are based on continuation techniques may not suffer from this drawback, but they do not guarantee to find all solutions as their number can change, see for instance [10], [11].

This paper proposes an alternative approach for investigating the load flow equations in power systems. Specifically, the problem of computing the solutions of these equations is firstly considered, using a strategy based on linear matrix inequalities (LMIs). In particular, this strategy requires to solve a convex optimization problem with LMI constraints, and guarantees to find all solutions provided that the dimension of a linear space obtained via the optimization is smaller than a known threshold. The benefit is hence that, whenever such a condition holds, one is guaranteed that no solution is lost. Moreover, it is explained that one can suitably repeat the proposed procedure, as this will facilitate the fulfillment of such a condition. Then, the problem of characterizing the set of admissible power is considered, in particular the computation of the largest power for which there exists a solution in the phases of the buses. While this problem amounts to solving a nonconvex optimization, it is shown in this paper that an upper bound of the sought power can be obtained via convex LMI optimization. Moreover, a necessary and sufficient condition is provided for establishing whether the found upper bound is tight or not. Some numerical examples illustrate the application and the benefits of the proposed approach.

The paper is organized as follows. Section II introduces the problem formulation and some preliminaries about the use of LMIs for investigating polynomial positivity. Section III describes the proposed approach for investigating the load flow equations. Section IV presents some numerical examples. Lastly, Section V reports some final remarks.

II. Preliminaries

A. Problem formulation

Let us start by introducing the notation adopted throughout the paper:

- $\mathbb{R}$: space of real numbers;
- 0_n: $n \times 1$ vector with all entries equal to 0;
- 1_n: $n \times 1$ vector with all entries equal to 1;
- I_n: $n \times n$ identity matrix;
- A': transpose of matrix A;
- $A > 0$ ($A \geq 0$): positive definite (positive semidefinite) matrix A;

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- $\|a\|$: 2-norm of vector a;
- s.t.: subject to.

Let us consider an interconnected power system with $n+1$ buses. The $(n+1)$st bus is a slack bus with specified voltage $V_{n+1} \in \mathbb{R}$ and phase angle $\theta_{n+1} = 0$. All the remaining buses are PV buses with specified voltage $V_i \in \mathbb{R}$ and (unspecified) phase angles $\theta_i \in \mathbb{R}$, $i = 1, \ldots, n$. Let $p_i \in \mathbb{R}$ be the net real power injected into the ith bus, and let us define the vectors

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \quad f(\theta) = \begin{pmatrix} f_1(\theta) \\ \vdots \\ f_n(\theta) \end{pmatrix} \tag{1}$$

where

$$f_i(\theta) = \sum_{j=1}^{n+1} V_i V_j Y_{i,j} \sin(\theta_i - \theta_j), \quad i = 1, \ldots, n \tag{2}$$

and $Y_{i,j} \in \mathbb{R}$ is the admittance of the lossless line joining buses i and j. Then, the load flow equation is (see for instance [1]):

$$p = f(\theta). \tag{3}$$

In this paper we address the following problems.

- Problem P1 (*load flow problem for PV buses*): to solve (3) in the unknown vector θ for a given vector p, i.e. to find the set

$$\Theta(p) = \{\theta \in [0, 2\pi)^n : \; p = f(\theta)\}. \tag{4}$$

- Problem P2 (*admissible power characterization for PV buses*): to characterize the set of admissible vectors p for which (20) has a solution in θ. In particular, we consider the determination of the maximum admissible power (2-norm) in this set, i.e.

$$\begin{array}{rcl} c_{max} & = & \max_{p \in \mathbb{R}^n} \|p\|_2 \\ & & \text{s.t.} \;\; \Theta(p) \neq \emptyset. \end{array} \tag{5}$$

Before proceeding it is worth noticing that, since the V_i are known and constant, they may be absorbed into the $Y_{i,j}$, and hence $f_i(\theta)$ in (2) can be simply rewritten as

$$f_i(\theta) = \sum_{j=1}^{n+1} Y_{i,j} \sin(\theta_i - \theta_j), \quad i = 1, \ldots, n. \tag{6}$$

Moreover, since $\theta_{n+1} = 0$ and since $\sin(\theta_i - \theta_j) = 0$ for all $i = j$, $f_i(\theta)$ in (6) boils down to

$$f_i(\theta) = \sum_{\substack{j=1 \\ j \neq i}}^{n} Y_{i,j} \sin(\theta_i - \theta_j) + Y_{i,n+1} \sin\theta_i, \quad i = 1, \ldots, n. \tag{7}$$

Lastly, we define the matrix $Y \in \mathbb{R}^{n+1 \times n+1}$ as

$$Y = (Y_{i,j})_{i,j=1,\ldots,n+1} \tag{8}$$

which is known to be a symmetric positive semidefinite matrix. Let us observe that the diagonal and the last row of Y do not affect the load flow equation (3).

B. Positive polynomials via LMIs

Positivity of a polynomial can be investigated via LMIs (linear matrix inequalities). Specifically, let $s(x)$ be a polynomial of degree $2m$ in $x \in \mathbb{R}^n$, and let $x^{\{m\}} \in \mathbb{R}^{\sigma(n,m)}$ be a vector containing all monomials of degree less than or equal to m in x, being $\sigma(n,m)$ the number of such monomials, which is given by

$$\sigma(n,m) = \frac{(n+m)!}{n!m!}. \tag{9}$$

For instance, one can select

$$x^{\{m\}} = \left(1, x_1, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^m\right)'. \tag{10}$$

Then, $s(x)$ can be written as

$$s(x) = x^{\{m\}'} S(\alpha) x^{\{m\}} \tag{11}$$

where $S(\alpha) \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$ is a symmetric affine linear matrix function expressed as

$$S(\alpha) = S + L(\alpha) \tag{12}$$

where $S \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$ is any symmetric matrix such that

$$s(x) = x^{\{m\}'} S x^{\{m\}}, \tag{13}$$

$L(\alpha)$ is a linear parametrization of the set

$$\begin{array}{rcl} \mathcal{L}(n,m) & = & \Big\{ L = L' \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)} : \\ & & x^{\{m\}'} L x^{\{m\}} = 0 \Big\}, \end{array} \tag{14}$$

and $\alpha \in \mathbb{R}^{\nu(n,m)}$ is a vector of free parameters, where the dimension $\nu(n,m)$ of $\mathcal{L}(n,m)$ is given by

$$\nu(n,m) = \frac{1}{2}\sigma(n,m)(\sigma(n,m)+1) - \sigma(n,2m). \tag{15}$$

This representation of polynomials via vector-matrix-vector products is known as the square matricial representation (SMR) [12] and Gram matrix method [13].

The expression of $s(x)$ in (11)–(12) was introduced in [12] in order to investigate positivity of polynomials. Indeed, the condition "$s(x)$ is positive" can be relaxed as "$s(x)$ is sum of squares of polynomials (SOS)", and this latter condition holds if and only if

$$\exists \alpha : \; S(\alpha) \geq 0. \tag{16}$$

The above condition is an LMI feasibility test, which can be solved through a convex optimization since the feasible set of an LMI is convex, see for example in [14]. SOS via LMI was proposed in [12]. The matrix S and the parametrization $L(\alpha)$ can be readily computed via simple algorithms, see for instance [15]. The SOS concept can be extended to the case of matrix polynomials, and SOS of matrix polynomials via LMI was proposed in [16], [17]. See also [18] and references therein regarding the conservatism of the SOS relaxation. The reader is also referred to [19] for details about the SMR and SOS polynomials.

Before proceeding it is useful to mention that optimizations with LMIs can be readily solved by using dedicated software such as [20], [21].

III. Proposed approach

In this section we describe the proposed solutions.

A. Solving problem P1 (load flow problem)

In order to solve (3) in the unknown vector θ, let us define the variables $x_i, y_i \in \mathbb{R}$, $i = 1, \ldots, n$, as

$$\begin{array}{rcl} x_i & = & \sin\theta_i, \quad i = 1\ldots, n+1 \\ y_i & = & \cos\theta_i, \quad i = 1\ldots, n+1. \end{array} \tag{17}$$

Clearly, since $\theta_{n+1} = 0$, one has $x_{n+1} = 0$ and $y_{n+1} = 1$. Moreover, since

$$\sin(\theta_i - \theta_j) = \sin\theta_i\cos\theta_j - \sin\theta_j\cos\theta_i, \tag{18}$$

it follows that $f_i(\theta)$ in (7) can be rewritten as

$$f_i(\theta) = \sum_{\substack{j=1\\ j\neq i}}^{n} Y_{i,j}(x_i y_j - x_j y_i) + Y_{i,n+1}x_i, \quad i = 1, \ldots, n. \tag{19}$$

Therefore, (3) boils down to the system of polynomial equations

$$\left\{ \begin{array}{rcl} p_i & = & \sum_{\substack{j=1\\ j\neq i}}^{n} Y_{i,j}(x_i y_j - x_j y_i) + Y_{i,n+1}x_i \\ 1 & = & x_i^2 + y_i^2 \\ i & = & 1, \ldots, n \end{array} \right. \tag{20}$$

The above system contains $2n$ equations of degree 2 in the $2n$ unknowns $x_1, \ldots, x_n, y_1, \ldots, y_n$.

Let us define the new variable $z \in \mathbb{R}^{2n}$ as

$$z = \left(\begin{array}{c} x \\ y \end{array} \right) \tag{21}$$

and the vectors

$$g(z) = \left(\begin{array}{c} g_1(z) \\ \vdots \\ g_n(z) \end{array} \right), \quad h(z) = \left(\begin{array}{c} h_1(z) \\ \vdots \\ h_n(z) \end{array} \right). \tag{22}$$

where

$$\begin{array}{rcl} g_i(z) & = & \sum_{\substack{j=1\\ j\neq i}}^{n} Y_{i,j}(x_i y_j - x_j y_i) + Y_{i,n+1}x_i, \\ & & i = 1, \ldots, n \\ h_i(z) & = & x_i^2 + y_i^2, \quad i = 1, \ldots, n. \end{array} \tag{23}$$

It follows that (20) can be rewritten as

$$\left\{ \begin{array}{rcl} p & = & g(z) \\ 1_n & = & h(z) \end{array} \right. \tag{24}$$

Let us define the set of solutions of (24) as

$$\mathcal{Z}(p) = \left\{ z \in \mathbb{R}^{2n} : \; p = g(z), \; 1_n = h(z) \right\}. \tag{25}$$

Clearly, the set $\mathcal{Z}(p)$ is related to the sought set $\Theta(p)$ by

$$\begin{array}{c} \theta \in \Theta(p) \\ \Updownarrow \\ t(\theta) \in \mathcal{Z}(p), \; \theta \in [0, 2\pi)^n \end{array} \tag{26}$$

where $t : \mathbb{R}^n \to \mathbb{R}^{2n}$ is the function

$$t(\theta) = (\sin\theta_1, \ldots, \sin\theta_n, \cos\theta_1, \ldots, \cos\theta_n)'. \tag{27}$$

In order to solve (24) in the unknown vector z, let us define

$$v(z) = \left(\|g(z) - p\|^2 + \|h(z) - 1_n\|^2 \right) u(z) \tag{28}$$

where $u(z)$ is any polynomial of degree 2δ satisfying $u(z) \geq 1$ for all z. In the sequel we will suppose for simplicity that $u(z) = \left(1 + \|z\|^2\right)^{\delta}$. Let us express $v(z)$ as described in Section II-B via

$$v(z) = z^{\{\delta+2\}'} V z^{\{\delta+2\}} \tag{29}$$

for some symmetric matrix $V = V' \in \mathbb{R}^{\sigma(2n,\delta+2)\times\sigma(2n,\delta+2)}$. Lastly, let us observe that the first equation in (24) can be rewritten as

$$p = A(x)y + b(x) \tag{30}$$

for some linear functions $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $b : \mathbb{R}^n \to \mathbb{R}^n$. The following result extends our previous result in [22] and provides a strategy for solving problem P1 through convex optimization with LMI constraints.

Theorem 1: Let $L(\alpha)$ be a linear parametrization of the set $\mathcal{L}(2n, \delta + 2)$ in (14). Let us define the optimization problem

$$\begin{array}{rcl} \gamma^* & = & \sup_{\gamma,\alpha} \gamma \\ & & \text{s.t.} \;\; V - \gamma I_{\sigma(2n,\delta+2)} + L(\alpha) > 0 \end{array} \tag{31}$$

and the matrix

$$M = V + L(\alpha^*) \tag{32}$$

where α^* is an optimal value of α in (31). Then,

$$\gamma^* > 0 \Rightarrow \Theta(p) = \emptyset. \tag{33}$$

Moreover,

$$\Theta(p) \neq \emptyset \Rightarrow \gamma^* = 0. \tag{34}$$

Lastly,

$$\begin{array}{c} \theta \in \Theta(p) \\ \Updownarrow \\ t(\theta)^{\{\delta+2\}} \in \ker(M), \; t(\theta) = \tau(x) \text{ for some } x \end{array} \tag{35}$$

where $\tau : \mathbb{R}^n \to \mathbb{R}^{2n}$ is the function

$$\tau(x) = \left(\begin{array}{c} x \\ A(x)^{-1}(p - b(x)) \end{array} \right). \tag{36}$$

Proof. Let us suppose $\gamma^* > 0$. Then, pre- and post-multiplying the first LMI in (31) by $z^{\{\delta+2\}'}$ and $z^{\{\delta+2\}}$ respectively, one gets

$$\begin{array}{rcl} 0 & < & z^{\{\delta+2\}'} \left(V - \gamma^* I_{\sigma(2n,\delta+2)} + L(\alpha^*) \right) z^{\{\delta+2\}} \\ & = & z^{\{\delta+2\}'} \left(V - \gamma^* I_{\sigma(2n,\delta+2)} \right) z^{\{\delta+2\}} \\ & < & z^{\{\delta+2\}'} V z^{\{\delta+2\}} \end{array} \tag{37}$$

which is equivalent to

$$\left(\|g(z) - p\|^2 + \|h(z) - 1_n\|^2 \right) u(z) > 0. \tag{38}$$

Since $u(z) > 0$ for all z, it follows that there does not exist z such that (24) holds. Consequently, (33) holds.

Next, let us suppose $\Theta(p) \neq \emptyset$. Clearly,

$$\gamma^* \ngtr 0 \quad (39)$$

because this would lead to a contradiction since (33) holds. Moreover, one has that

$$\gamma^* \geq 0 \quad (40)$$

since V can be constructed positive semidefinite according to

$$V = K' \left((G'G + H'H) \otimes I_{\delta(n+1)} \right) K \quad (41)$$

where $G \in \mathbb{R}^{n \times \sigma(2n,2)}$ and $H \in \mathbb{R}^{n \times \sigma(2n,2)}$ are the matrices fulfilling

$$\begin{array}{rcl} g(z) - p & = & Gz^{\{2\}} \\ h(z) - 1_n & = & Hz^{\{2\}}, \end{array} \quad (42)$$

and K is the matrix fulfilling

$$z^{\{2\}} \otimes \underbrace{z^{\{1\}} \otimes \cdots \otimes z^{\{1\}}}_{\delta \text{ times}} = Kz^{\{\delta+2\}}. \quad (43)$$

Hence, for any possible choice of V there exists α such that $V + L(\alpha) \geq 0$, and therefore (34) holds.

Lastly, let us observe that $\theta \in \Theta(p)$ if and only if (24) holds with $z = t(\theta)$. Moreover, (24) holds if and only if

$$\|g(z) - p\|^2 + \|h(z) - 1_n\|^2 = 0. \quad (44)$$

If $\Theta(p) \neq \emptyset$, then $\gamma^* = 0$ from (34), and hence

$$\left(\|g(z) - p\|^2 + \|h(z) - 1_n\|^2 \right) u(z) = z^{\{\delta+2\}'} M z^{\{\delta+2\}}. \quad (45)$$

We have that $u(z) > 0$ for all z, moreover

$$M \geq 0 \quad (46)$$

due to the LMI in (31). This implies that

$$z^{\{\delta+2\}'} M z^{\{\delta+2\}} = 0 \iff z^{\{\delta+2\}} \in \ker(M) \quad (47)$$

and hence from (26) and (30) one has that (35) holds. □

Theorem 1 provides a strategy for finding $\Theta(p)$ through the optimization (31). In particular, it is explained that $\Theta(p)$ is empty whenever $\gamma^* > 0$. Moreover, if this latter condition does not hold, then necessarily $\gamma^* = 0$, and all the solutions in $\Theta(p)$ can be found by looking for suitable vectors in $\ker(M)$.

As explained for instance in [22], [23], this last operation amounts to performing linear algebra operations provided that the dimension of $\ker(M)$ is smaller than a known threshold. Hence, whenever such a condition holds, one is guaranteed to find all solutions in $\Theta(p)$.

Let us observe that the free variables in the vectors to be searched into $\ker(M)$ are n since the other n variables are given by the equation $t(\theta) = \tau(x)$ in (35), and this allows one to double the allowed threshold for the above condition (in this respect, let us observe that $A(x)$ is invertible whenever $\Theta(p)$ is finite). Lastly, the role of the polynomials $u(z)$ is to facilitate the fulfillment of this condition: indeed, by increasing δ, one also increases this threshold while the dimension of $\ker(M)$ is expected to remain constant being related to the number of solutions in $\Theta(p)$.

B. Solving problem P2 (admissible powers set)

In this section we address the computation of c_{max} in (5). As it can be observed, c_{max} represents the radius of the smallest sphere with center in the origin containing the set of admissible powers, i.e. the the set of admissible vectors p for which (20) has a solution in θ.

Let $q_1(z), \ldots, q_n(z) \in \mathbb{R}$ be variable polynomials of degree 2δ, and let us define the vector of polynomials

$$q(z) = (q_1(z), \ldots, q_n(z))'. \quad (48)$$

From $q(z)$ we define the polynomial

$$r(z) = q(z)'(1_n - h(z)). \quad (49)$$

It follows that $r(z)$ is a polynomial of degree $2(\delta + 1)$.

Let us define

$$m = \max\{\delta + 1, 2\}. \quad (50)$$

Let us express $g(z)$ and $q(z)$ as

$$\begin{array}{rcl} g(z) & = & Gz^{\{m\}} \\ q(z) & = & qz^{\{2\delta\}} \end{array} \quad (51)$$

where $G \in \mathbb{R}^{n \times \sigma(2n,m)}$ and $q \in \mathbb{R}^{\sigma(2\delta)}$. Moreover, let us express $r(z)$ as

$$r(z) = z^{\{m\}'} R(q) z^{\{m\}} \quad (52)$$

for some $R(q) = R(q)' \in \mathbb{R}^{\sigma(2n,m) \times \sigma(2n,m)}$. Let us observe that $R(q)$ depends linearly on q. Lastly, let us express the constant polynomial 1 as

$$1 = z^{\{m\}'} O z^{\{m\}} \quad (53)$$

for some $O = O' \in \mathbb{R}^{\sigma(2n,m) \times \sigma(2n,m)}$. The following result provides a strategy for computing an upper bound of c_{max} via a convex LMI optimization based on Theorem 1 and Stengle's Positivstellensatz [24].

Theorem 2: Let $L(\alpha)$ be a linear parametrization of the set $\mathcal{L}(2n, m)$ in (14). Let us define

$$c^* = \sqrt{\bar{c}^*} \quad (54)$$

where $\bar{c}^*$ is the solution of the optimization problem

$$\begin{array}{rcl} \bar{c}^* & = & \inf\limits_{\bar{c}, q, \alpha} \bar{c} \\ & & \text{s.t. } \bar{c}O + R(q) + L(\alpha) - G'G > 0 \end{array} \quad (55)$$

Then,

$$c^* \geq c_{max}. \quad (56)$$

Proof. Let us suppose that the LMI in (55) holds for some $\bar{c}$, q, and α. Then, pre- and post-multiplying this LMI by $z^{\{m\}'}$ and $z^{\{m\}}$ respectively, one gets

$$\begin{array}{rcl} 0 & < & z^{\{m\}'} \left(\bar{c}O + R(q) + L(\alpha) - G'G \right) z^{\{m\}} \\ & = & z^{\{m\}'} \left(\bar{c}O + R(q) - G'G \right) z^{\{m\}} \\ & = & \bar{c} + r(z) - \|g(z)\|^2 \end{array} \quad (57)$$

i.e.

$$\bar{c} + r(z) > \|g(z)\|^2. \quad (58)$$

Let us suppose now that z is admissible, i.e. such that $h(z) = 1_n$. We have that

$$h(z) = 1_n \Rightarrow r(z) = 0 \tag{59}$$

and hence

$$h(z) = 1_n \Rightarrow \bar{c} > \|g(z)\|^2. \tag{60}$$

Therefore, (56) holds. □

Theorem 2 provides an upper bound of c_{max} for any chosen δ, i.e. the degree of $q(z)$. It can be observed that the conservatism of this upper bound can be decreased by increasing δ.

A natural question arises: is the found upper bound $\hat{c}$ tight, i.e. $\hat{c} = c^*$? The following result provides a necessary and sufficient condition for establish whether the found upper bound $\hat{c}$ coincides with the sought c^*.

Theorem 3: Let us define the matrix

$$N = \bar{c}^* O + R(q^*) + L(\alpha^*) - G'G. \tag{61}$$

where $\bar{c}^*, q^*, \alpha^*$ are the optimal values of $\bar{c}, q, \alpha$ in (55). Then, $c^* = c_{max}$ if and only if there exists $\theta \in \mathbb{R}^n$ such that

$$t(\theta)^{\{m\}} \in \ker(N),\ t(\theta) = \tau(x) \text{ for some } x \tag{62}$$

and

$$c^* = \|f(\theta)\|. \tag{63}$$

Proof. "$\Leftarrow$" Let us suppose $c^* = c_{max}$. Then, there clearly exists $\theta \in \mathbb{R}^n$ such that (63) holds. By defining $z = t(\theta)$, this implies

$$\bar{c}^* - \|g(z)\|^2 = 0. \tag{64}$$

Pre- and post-multiplying N by $z^{\{m\}'}$ and $z^{\{m\}}$ respectively one gets

$$\begin{array}{rcl} z^{\{m\}'} N z^{\{m\}} & = & z^{\{m\}'} \left(\bar{c}^* O + R(q^*) - G'G\right) z^{\{m\}} \\ & = & \bar{c}^* + r(z) - \|g(z)\|^2 \\ & = & \bar{c}^* - \|g(z)\|^2. \end{array} \tag{65}$$

Therefore,

$$z^{\{m\}'} N z^{\{m\}} = 0. \tag{66}$$

Since N is positive semidefinite, it hence follows that (62) holds with $t(\theta) = z$.

"$\Rightarrow$" Let us suppose that (62)–(63) hold. Then, $c^* \geq c_{max}$ from (56), moreover $c^* \leq c_{max}$ from (63). Therefore, $c^* = c_{max}$. □

Theorem 3 provides a necessary and sufficient condition for establishing whether the upper bound c^* is tight or not. This condition amounts to looking for suitable vectors in $\ker(N)$, which can be performed as described at the end of Section III-A.

IV. Examples

A. Example 1

Let us consider problem P1, i.e. computing the set $\Theta(p)$ in (4), in the numerical case

$$p = \begin{pmatrix} 0.3 \\ 0.4 \\ 0.5 \end{pmatrix}, \quad Y = \begin{pmatrix} \star & 1 & 0.7 & 0.4 \\ 1 & \star & 0.3 & 0.8 \\ 0.7 & 0.3 & \star & 0.6 \\ \star & \star & \star & \star \end{pmatrix}$$

(the symbol "$\star$" indicates components of Y that do not affect (3)).

Let us use Theorem 1 with the simple choice $\delta = 0$. We find that $\gamma^* = 0$, moreover the dimension of $\ker(M)$ is 2, which is smaller than the allowed threshold (5 for this case). This ensures that all solutions can be found, in particular by proceeding as described at the end of Section III-A we find that the set $\mathcal{Z}(p)$ in (25) is

$$\mathcal{Z}(p) = \left\{ \begin{pmatrix} 0.6610 \\ 0.7493 \\ 0.5603 \\ -0.7504 \\ -0.6622 \\ -0.8283 \end{pmatrix}, \begin{pmatrix} 0.6764 \\ 0.6264 \\ 0.7138 \\ 0.7366 \\ 0.7795 \\ 0.7004 \end{pmatrix} \right\}.$$

From $\mathcal{Z}(p)$, we directly obtain $\Theta(p)$ according to (26):

$$\Theta(p) = \left\{ \begin{pmatrix} 2.4195 \\ 2.2946 \\ 2.5469 \end{pmatrix}, \begin{pmatrix} 0.7429 \\ 0.6770 \\ 0.7949 \end{pmatrix} \right\}.$$

For comparison, we attempt to solve (24) by using analytical methods (which, to our best knowledge, are the only methods ensuring to find all solutions), in particular via the function "Solve" of Matlab. It is interesting to observe that the required computational time is 69.0 s, which is significantly larger than that required by the proposed approach (1.6 s).

Next, we consider problem P2, i.e. to compute c_{max} in (5). From Theorem 2 we find (with $\delta = 1$) the upper bound

$$c^* = 2.8529$$

By using Theorem 3 we also establish that this upper bound is tight, and hence

$$c_{max} = 2.8529$$

(the computational time is 1.6 s). In particular, the optimal value of p in (5) is given by

$$p_{max} = p^* = \begin{pmatrix} 1.8561 \\ -1.7882 \\ -1.2233 \end{pmatrix}.$$

B. Example 2

In this example we consider the numerical case

$$p = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}, \quad Y = \begin{pmatrix} \star & 0.6 & 0.8 & 1 & 0.5 \\ 0.6 & \star & 1 & 1.2 & 0.5 \\ 0.8 & 1 & \star & 1.4 & 0.5 \\ 1 & 1.2 & 1.4 & \star & 0.5 \\ \star & \star & \star & \star & \star \end{pmatrix}.$$

Let us proceed as in Example 1. We have that the dimension of $\ker(M)$ is smaller than the allowed threshold ($2 < 6$ for this case), which ensures that all solutions can be found. In particular we obtain

$$\mathcal{Z}(p) = \left\{ \begin{pmatrix} 0.6274 \\ 0.5959 \\ 0.5966 \\ 0.5801 \\ -0.7787 \\ -0.8031 \\ -0.8025 \\ -0.8143 \end{pmatrix}, \begin{pmatrix} 0.5784 \\ 0.6029 \\ 0.6024 \\ 0.6162 \\ 0.8157 \\ 0.7978 \\ 0.7982 \\ 0.7875 \end{pmatrix} \right\}.$$

From $\mathcal{Z}(p)$, we directly obtain $\Theta(p)$ according to (26):

$$\Theta(p) = \left\{ \begin{pmatrix} 2.4634 \\ 2.5033 \\ 2.5023 \\ 2.5226 \end{pmatrix}, \begin{pmatrix} 0.6168 \\ 0.6471 \\ 0.6466 \\ 0.6640 \end{pmatrix} \right\}.$$

Again, we compare our approach with analytical methods as done in Example 1. We find that, after waiting more than one day, we cannot obtain the set $\mathcal{Z}(p)$. Instead, the computational time of the proposed approach is 7.8 s.

For problem P2 we find from Theorem 2 the upper bound $c^* = 4.8780$ which turns out to be tight by using Theorem 3, i.e. $c_{max} = 4.8780$ (the computational time is 10.0 s). In particular, the optimal value of p in (5) is given by

$$p_{max} = p^* = \begin{pmatrix} 0.7614 \\ -1.8543 \\ -2.6846 \\ 3.5454 \end{pmatrix}.$$

V. CONCLUSION

We have considered the computation of the solutions of the load flow equations in power systems. Although various methods have been proposed, the problem is still open since analytical methods can be used only for very small systems, and non-analytical methods cannot guarantee to find all solutions. The proposed approach ensures to find all solutions provided that the dimension of a linear space provided by the optimization is smaller than a known threshold. Hence, whenever such a condition holds, one is guaranteed that no solution is lost. Then, we have considered the characterization of the set of admissible power, in particular the computation of the largest admissible power. While this problem amounts to solving a nonconvex optimization, in this paper we have proposed the computation of an upper bound via a convex LMI optimization. Moreover, we have proposed a necessary and sufficient condition which allows one to establish whether the upper bound is tight or not.

Future work will investigate large-scale models involving also reactive power and variable voltages.

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