A Hyperbolic Lindstedt-Poincaré Method for Homoclinic Motion of a Kind of Strongly Nonlinear Autonomous Oscillators

Y. Y. Chen • S. H. Chen • K. Y. Sze

Abstract A hyperbolic Lindstedt–Poincaré method is presented to determine the homoclinic solutions of a kind of nonlinear oscillators, in which critical value of the homoclinic bifurcation parameter can be determined. The generalized Liénard oscillator is studied in detail, and the present method's predictions are compared with those of Runge–Kutta method to illustrate its accuracy.

Keywords Lindstedt-Poincaré Method, Hyperbolic function, Nonlinear autonomous oscillator, Homoclinic orbit

1 Introduction

In the last few decades, many new techniques have been presented for obtaining periodic solution of the nonlinear oscillator equation in the form of

$$\ddot{x} + g(x) = \varepsilon f(x,\dot{x}), \tag{1}$$

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where $g(x)$ and $f(x,\dot{x})$ are nonlinear functions of their arguments and ε is a small positive parameter. With reference to the periodic functions employed in the solutions, these techniques can be categorized into the circular (trigonometric) function perturbation procedures, the elliptic function perturbation procedures and the generalized harmonic function perturbation procedures as described in [1]. Furthermore, much effort has been paid to investigate the stability and bifurcation of periodic solution. For example,Wang and Hu[2] presented a modified averaging scheme with application to the secondary Hopf bifurcation of a delayed van der Pol oscillator. Gan and He [3] studied the structural safety in a kind of excited Duffing oscillator.

In most cases, the homoclinic (heteroclinic) orbit is a separatrix of the periodic solutions and the non-periodic solutions of a nonlinear dynamic system, and thus plays an important role in studying the global bifurcation of nonlinear systems and attracts considerable attention. Xu et al. [4] presented a perturbation-incremental method to study the separatrices and the limit cycles of strongly nonlinear oscillators, which was used by Chan et al. [5] and Chen et al. [6] to study the stability and the bifurcations of limit cycles and the semi-stable limit cycles, respectively. Moreover, Zhang and Lu [7] presented a frequency-incremental method to study the homoclinic bifurcation of strongly nonlinear oscillators, and Zhang et al. [8] applied the undetermined fundamental frequency method to predict the heteroclinic bifurcation of strongly nonlinear oscillators. Belhaq [9] and his coworkers presented an analytical method to predict homoclinic bifurcation of autonomous oscillators, in which the homoclinic bifurcation value was determined by considering the period of the limit cycles which approaches infinity. To improve the accuracy of their technique, they also employed the elliptic averaging method [10] and the elliptic Linstedts-Poincaré method [11] which lead to the same results as the standard Melnikov method. It isworth noting that the afore-mentioned techniques are based on the periodic functions and, thus, cannot yield the homoclinic (heteroclinic) solution. However, the present authors have recently proposed a hyperbolic perturbation method to determine homoclinic orbits of some strongly nonlinear autonomous oscillators [1].

Based on the previous work, a hyperbolic Lindstedt–Poincaré method is presented in this paper to determine the homoclinic orbits of certain nonlinear autonomous oscillators, in which hyperbolic functions, rather than the usual periodic functions, are employed in the classical Lindstedt–Poincaré procedure and the critical value of the homoclinic bifurcation parameter μ can be thereby determined. To show the essences of the present method, the typical generalized Liénard equation is studied in detail, and comparison is made between the results predicted by Runge–Kutta method and the present method. It can be seen that the present method attains fairly good accuracy even for amoderately large value of ε.

2 The hyperbolic Lindstedt-Poincaré method

To demonstrate the hyperbolic Lindstedt-Poincaré method, the following nonlinear autonomous system is considered:

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon f(\mu, x, \dot{x}), \tag{2}$$

in which μ is the homoclinic bifurcation parameter. Its critical value μ_c, under which there exist a homoclinic solution, will be determined in a later procedure. Let

$$x = x_0 + \varepsilon x_1 + \cdots = \sum_{n=0}^{+\infty} \varepsilon^n x_n . \tag{3}$$

Expansions of μ_c and $f(\mu_c, x, \dot{x})$ with respect to ε can be expressed as:

$$\mu_c = \mu_{c0} + \varepsilon \mu_{c1} + \cdots = \sum_{n=0}^{\infty} \varepsilon^n \mu_{cn} , \tag{4}$$

$$\begin{aligned} f(\mu_c, x, \dot{x}) &= f(\mu_{c0}, x_0, \dot{x}_0) + \varepsilon[\mu_{c1} f,_x(\mu_{c0}, x_0, \dot{x}_0) \\ &\quad + x_1 f,_x(\mu_{c0}, x_0, \dot{x}_0) + \dot{x}_1 f,_{\dot{x}}(\mu_{c0}, x_0, \dot{x}_0)] \\ &\quad + \varepsilon^2[\mu_{c2} f,_\mu(\mu_{c0}, x_0, \dot{x}_0) + x_2 f,_x(\mu_{c0}, x_0, \dot{x}_0) \\ &\quad + \dot{x}_2 f,_{\dot{x}}(\mu_{c0}, x_0, \dot{x}_0) + \mu_{c1} x_1 f,_{\mu x}(\mu_{c0}, x_0, \dot{x}_0) \\ &\quad + \mu_{c1} \dot{x}_1 f,_{\mu \dot{x}}(\mu_{c0}, x_0, \dot{x}_0) + x_1 \dot{x}_1 f,_{x\dot{x}}(\mu_{c0}, x_0, \dot{x}_0) \\ &\quad + \frac{1}{2}\mu_{c1}^2 f,_{\mu\mu}(\mu_{c0}, x_0, \dot{x}_0) + \frac{1}{2} x_1^2 f,_{xx}(\mu_{c0}, x_0, \dot{x}_0) \\ &\quad + \frac{1}{2}\dot{x}_1^2 f,_{\dot{x}\dot{x}}(\mu_{c0}, x_0, \dot{x}_0)] + \cdots \\ &= \sum_{n=0}^{+\infty} \frac{\varepsilon^n}{n!} \left. \frac{\mathrm{d}^n f(\mu_c(\varepsilon), x(\varepsilon), \dot{x}(\varepsilon))}{\mathrm{d}\varepsilon^n} \right|_{\varepsilon=0} \end{aligned} \tag{5}$$

Where $f,_\mu = \partial f/\partial \mu$, $f,_{\mu x} = \partial^2 f/(\partial \mu \partial x)$, etc. By substituting Eqs. (3), (4) and (5) into Eq. (2) and comparing coefficients of ε, we have

$$\varepsilon^0: \quad \ddot{x}_0 + c_1 x_0 + c_2 x_0^2 = 0; \tag{6}$$

$$\varepsilon^1: \quad \ddot{x}_1 + (c_1 + 2c_2 x_0) x_1 = f(\mu_{c0}, x_0, \dot{x}_0); \tag{7}$$

$$\begin{aligned} \varepsilon^2: \quad \ddot{x}_2 + (c_1 + 2c_2 x_0) x_2 &= \mu_{c1} f,_\mu(\mu_{c0}, x_0, \dot{x}_0) \\ &\quad + x_1 f,_x(\mu_{c0}, x_0, \dot{x}_0) + \dot{x}_1 f,_{\dot{x}}(\mu_{c0}, x_0, \dot{x}_0) \\ &\quad - c_2 x_1^2; \end{aligned} \tag{8}$$

$$\vdots$$

$$\begin{aligned} \varepsilon^n: \quad \ddot{x}_n + (c_1 + 2c_2 x_0) x_n, \\ = \frac{1}{(n-1)!} \left. \frac{\mathrm{d}^{n-1} f(\mu_c(\varepsilon), x(\varepsilon), \dot{x}(\varepsilon))}{\mathrm{d}\varepsilon^{n-1}} \right|_{\varepsilon=0} \\ - c_2 \sum_{i=1}^{n-1} x_{n-i} x_i, \quad (n \geq 2); \end{aligned} \tag{9}$$

$$\vdots$$

Equation (6) has an exact analytical homoclinic solution in terms of hyperbolic functions [1]. The solution is

$$x_0 = a\,\mathrm{sech}^2\,\omega_0 t + b\,, \tag{10a}$$

$$\dot{x}_0 = -2a\omega_0\,\mathrm{sech}^2\,\omega_0 t\tanh\omega_0 t\,. \tag{10b}$$

$$\ddot{x}_0 = 2a\omega_0^2(3\tanh^2\omega_0 t - 1)\,\mathrm{sech}^2\,\omega_0 t\,, \tag{11}$$

where

$$\omega_0^2 = |c_1|/4\,, \tag{12a}$$

$$a = 3|c_1|/2c_2\,, \tag{12b}$$

$$b = -(|c_1| + c_1)/2c_2\,. \tag{12c}$$

It is trivial to prove that Eqs. (10) and (11) satisfy Eq. (2). As [12]

$$\mathrm{sech}(\pm\infty) = \tanh(0) = 0\,, \tag{13a}$$

$$\mathrm{sech}(0) = \tanh(+\infty) = 1\,, \tag{13b}$$

$$\tanh(-\infty) = -1\,, \tag{13c}$$

we have

$$x_0(0) = a + b\,, \tag{14a}$$

$$x_0(\pm\infty) = b\,, \tag{14b}$$

$$\dot{x}_0(\pm\infty) = \ddot{x}_0(\pm\infty) = 0\,, \tag{14c}$$

$$\dot{x}_0(0) = 0\,. \tag{14d}$$

Hence, Eqs. (14b,c) show that the homoclinic solution approaches the saddle point (b, 0) in phase plane as time $t \to \pm\infty$.

Differentiating Eq. (6) with respect to t leads to

$$\dddot{x}_0 + c_1\dot{x}_0 + 2c_2 x_0\dot{x}_0 = 0\,. \tag{15}$$

It can be seen from Eq. (15) that $\dot{x}_0$ is a solution of the homogeneous part of Eq. (7). From the theory of linear differential equations, the particular solution of Eq. (7) can be expressed as:

$$x_1 = \dot{x}_0\int\frac{1}{\dot{x}_0^2}\left[\int\dot{x}_0 f(\mu_{c0}, x_0, \dot{x}_0)\,\mathrm{d}t\right]\mathrm{d}t\,. \tag{16}$$

Since we are concerned only with the homoclinic solution which is independent of the initial conditions, the initial conditions and the homogeneous solution of $x1$ are ignored. This practice is similar to that of the classical Lindstedt–Poincaré procedure for periodic solutions of autonomous oscillators [13].

As a homoclinic solution, $(x, \dot{x})$ approaches a saddle point in the phrase plane as time $t \to \pm\infty$. Thus, x_1 and $\dot{x}_1$ should be bounded as $t \to \pm\infty$. Mathematically,

$$x_1(\pm\infty) \neq \pm\infty\,, \tag{17a}$$

$$\dot{x}_1(\pm\infty) \neq \pm\infty\,. \tag{17b}$$

By multiplying both sides of Eq. (7) with $\dot{x}_0$ and integrating the equation from -∞ to +∞, we have

$$(\dot{x}_0\dot{x}_1 - \ddot{x}_0 x_1)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}(\dddot{x}_0 + c_1\dot{x}_0 + 2c_2 x_0\dot{x}_0)x_1\,\mathrm{d}t\,. = \int_{-\infty}^{+\infty}\dot{x}_0 f(\mu_{c0}, x_0, \dot{x}_0)\,\mathrm{d}t \tag{18}$$

With Eq. (15) invoked, the above equation is reduced to

$$(\dot{x}_0\dot{x}_1 - \ddot{x}_0 x_1)\Big|_{-\infty}^{+\infty} = \int_{-\infty}^{+\infty}\dot{x}_0 f(\mu_{c0}, x_0, \dot{x}_0)\,\mathrm{d}t\,. \tag{19}$$

From Eqs. (14c) and (17), the left hand side of Eq. (19) must vanish, i.e.

$$\int_{-\infty}^{+\infty}\dot{x}_0 f(\mu_{c0}, x_0, \dot{x}_0)\,\mathrm{d}t = 0\,. \tag{20}$$

Therefore, Eq. (20) is a necessary condition for Eq. (17a, b). One can conveniently determine the critical value of $\mu 0$ by solving Eq. (20) and consequently avoid secular terms of $x1$ in the later procedure. In an alternative yet more cumbersome way, one can ignore Eq. (20) but determine $\mu 0$ by eliminating the secular terms in $x1$. This procedure will be illustrated more clearly by an example involving a generalized Liénard oscillator in the next section.

Multiplying $\dot{x}_0$ to Eq. (9) and integrating the equation from -∞ to +∞ give

$$(\dot{x}_0\dot{x}_2 - \ddot{x}_0 x_2)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}(\dddot{x}_0 + c_2\dot{x}_0 + 2c_2 x_0\dot{x}_0)x_2\,\mathrm{d}t$$
$$= \int_{-\infty}^{+\infty}\dot{x}_0[\mu_{c1} f_{,\mu}(\mu_{c0}, x_0, \dot{x}_0) + x_1 f_{,x}(\mu_{c0}, x_0, \dot{x}_0)$$

$$+\dot{x}_1 f_{,\dot{x}}(\mu_{c0}, x_0, \dot{x}_0) - c_2 x_1^2]\,\mathrm{d}t \,. \qquad (21)$$

By recalling Eq. (15), Eq. (21) is reduced as

$$(\dot{x}_0\dot{x}_2 - \ddot{x}_0 x_2)\Big|_{-\infty}^{+\infty} = \int_{-\infty}^{+\infty} \dot{x}_0[\mu_{c1} f_{,\mu}(\mu_{c0}, x_0, \dot{x}_0)$$
$$+x_1 f_{,x}(\mu_{c0}, x_0, \dot{x}_0) + \dot{x}_1 f_{,\dot{x}}(\mu_{c0}, x_0, \dot{x}_0) - c_2 x_1^2]\,\mathrm{d}t \,. \qquad (22)$$

From Eq. (14c) and the conditions

$$x_2(\pm\infty) \neq \pm\infty \,, \qquad (23a)$$
$$\dot{x}_2(\pm\infty) \neq \pm\infty \,, \qquad (23b)$$

$$\int_{-\infty}^{+\infty} \dot{x}_0[\mu_{c1} f_{,\mu}(\mu_{c0}, x_0, \dot{x}_0) + x_1 f_{,x}(\mu_{c0}, x_0, \dot{x}_0)$$
$$+\dot{x}_1 f_{,\dot{x}}(\mu_{c0}, x_0, \dot{x}_0) - c_2 x_1^2]\,\mathrm{d}t = 0 \,. \qquad (24)$$

Hence, the value of μ_{c1} under which there exists a homoclinic solution of Eq. (2) can be determined by Eq. (24). Consequently, one can eliminate the secular terms of the 2nd order solution

$$x_2 = \dot{x}_0 \int \frac{1}{\dot{x}_0^2} \Big\{ \int \dot{x}_0[\mu_{c1} f_{,\mu}(\mu_{c0}, x_0, \dot{x}_0)$$
$$+x_1 f_{,x}(\mu_{c0}, x_0, \dot{x}_0) + \dot{x}_1 f_{,\dot{x}}(\mu_{c0}, x_0, \dot{x}_0)$$
$$-c_2 x_1^2]\,\mathrm{d}t \Big\}\,\mathrm{d}t \qquad (25)$$

It is worth pointing out that Eq. (20) can also be derived to approximate μ_c by the standard Melnikov method, Belhaq's technique combined with elliptic solutions [10, 11] or the hyperbolic perturbation method [1]. Using different methods, here we expand μc in the form of Eq. (4) which can lead to a higher order approximation for critical value of μc. Furthermore, the explicit homoclinic solution can also be approximately constructed as the procedure is based on hyperbolic functions instead of periodic functions.

Similarly, one can determine the value of $\mu_c(n-1)$ by solving

$$\int_{-\infty}^{+\infty} \dot{x}_0 \left\{ \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1} f(\mu_c(\varepsilon), x(\varepsilon), \dot{x}(\varepsilon))}{\mathrm{d}\varepsilon^{n-1}} \bigg|_{\varepsilon=0} \right.$$
$$\left. -c_2 \sum_{i=1}^{n-1} x_{n-i} x_i \right\} \mathrm{d}t = 0 \,. \qquad (26)$$

Consequently, one can eliminate the secular terms of the n-th order solution

$$x_n = \dot{x}_0 \int \frac{1}{\dot{x}_0^2} \left\{ \int \dot{x}_0 [\frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1} f(\mu_c(\varepsilon), x(\varepsilon), \dot{x}(\varepsilon))}{\mathrm{d}\varepsilon^{n-1}} \bigg|_{\varepsilon=0} \right.$$
$$\left. -c_2 \sum_{i=1}^{n-1} x_{n-i} x_i]\,\mathrm{d}t \right\} \mathrm{d}t \,. \qquad (27)$$

However, the procedure would be increasingly cumbersome as the solution order goes up. More importantly, the computational results show that the solution up to the order εx_1 is fairly accurate even for the moderately large parameter ε and the homoclinic bifurcation parameter curves predicted by the present method show a higher accuracy than those obtained by periodic solutions.

In the classical Lindstedt-Poincaré procedure for periodic solution, one expands the nonlinear frequency with respect to ε and determines the frequency by eliminating secular terms. In the present procedure for homoclinic solution, we expand the homoclinic bifurcation critical parameter μ_c with respect to ε and similarly determines the critical parameter by eliminating the secular terms. In this light, the present method can be regarded as a hyperbolic Lindstedt-Poincaré method for homoclinic solutions of nonlinear oscillators.

3 A study of the generalized Liénard oscillator

As a sample application of the present method, the following generalized Liénard equation is considered:

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon(\mu + \mu_1 x - \mu_2 x^2 - \mu_3 \dot{x}^2)\dot{x} \,. \qquad (28)$$

In other words,

$$f(\mu, x, \dot{x}) = \dot{x}(\mu + \mu_1 x - \mu_2 x^2 - \mu_3 \dot{x}^2) \,, \qquad (29)$$

in which μ_1 and μ_2 are constants whereas μ is considered as a homoclinic bifurcation parameter. Let

$$
\begin{aligned}
I(t) &= \int \dot{x}_0 f(\mu_{c0}, x_0, \dot{x}_0)\,\mathrm{d}t \\
&= \int \dot{x}_0^2(\mu_{c0} + \mu_1 x_0 - \mu_2 x_0^2 - \mu_3 \dot{x}_0)\,\mathrm{d}t . \qquad (30)
\end{aligned}
$$

With x_0 and $\dot{x}_0$ in Eqs. (10a,b) substituted into Eq. (30), the latter equation becomes

$$
\begin{aligned}
I(t) = 4a^2\omega_0(&A_1 + A_2 \operatorname{sech}^2 \omega_0 t \\
&+A_3 \operatorname{sech}^4 \omega_0 t + A_4 \operatorname{sech}^6 \omega_0 t)\tanh^3 \omega_0 t \\
&+aA_5(1-3\tanh^2 \omega_0 t)\operatorname{sech}^2 \omega_0 t , \qquad (31)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \frac{2}{15}\mu_{c0} + (\frac{8}{105}a + \frac{2}{15}b)\mu_1 - (\frac{16}{315}a^2 + \frac{2}{15}b^2 \\
&\quad + \frac{16}{105}ab)\mu_2 , \qquad (32a)\\
A_2 &= \frac{3}{2}A_1 , \qquad (32b)\\
A_3 &= \frac{1}{7}a\mu_1 - (\frac{2}{7}ab + \frac{2}{21}a^2)\mu_2 , \qquad (33a)\\
A_4 &= -\frac{1}{9}a^2\mu_2 , \qquad (33b)\\
A_5 &= -\frac{1}{6}a\mu_3 . \qquad (33c)
\end{aligned}
$$

Thus, Eq. (20) or $I(t)\big|_{-\infty}^{+\infty} = 0$ yields

$$
\begin{aligned}
&4a^2\omega_0[(A_1 + A_2 \operatorname{sech}^2 \omega_0 t + A_3 \operatorname{sech}^4 \omega_0 t \\
&\quad +A_4 \operatorname{sech}^6 \omega_0 t)\tanh^3 \omega_0 t \\
&\quad +A_5\omega_0(2-\frac{3}{2}\operatorname{sech}^2 \omega_0 t)\operatorname{sech}^6 \omega_0 t]\Big|_{-\infty}^{+\infty} \\
&= 8a^2\omega_0 A_1 = 0 . \qquad (34)
\end{aligned}
$$

For non-zero a and ω_0,

$$A_1 = 0 \qquad (35)$$

By virtue of Eq. (32b), the above equation implies

$$A_2 = 0 . \qquad (36)$$

From the definition of A_1 in Eq. (32a),

$$\mu_{c0} = -(\frac{4}{7}a + b)\mu_1 + (\frac{8}{21}a^2 + b^2 + \frac{8}{7}ab\ )\mu_2 . \qquad (37)$$

From which the value of parameter μ_{c0} can be determined. Then, Eq. (16) becomes

$$x_1 = \dot{x}_0 \int \frac{1}{\dot{x}_0^2} I(t)\,\mathrm{d}t . \qquad (38)$$

After substituting Eq. (31) into Eq. (38), integration leads to

$$
\begin{aligned}
x_1 = a\operatorname{sech}^2 \omega_0 t &\Big\{-\frac{1}{\omega}[2A_3 \ln(\cosh \omega_0 t) \\
&+ A_4 \tanh^2 \omega_0 t]\tanh \omega_0 t + A_5(1-3\tanh^2 \omega_0 t)\Big\} \\
&+\frac{a}{\omega_0}[(\frac{1}{2}A_1 + A_2)\operatorname{sech}^2 \omega_0 t \tanh \omega_0 t \\
&-2A_2 \tanh \omega_0 t - A_1 \sinh 2\omega_0 t] . \qquad (39)
\end{aligned}
$$

As $\sinh 2\omega_0 t$ tends to infinity as $t \to \infty$, $A_1 \sinh 2\omega_0 t$ is a secular term for homoclinic solution. Hence, A_1 should vanish and this leads to Eq. (36). Consequently, the first order solution becomes

$$
\begin{aligned}
x_1 = a\operatorname{sech}^2 \omega_0 t &\Big\{-\frac{1}{\omega}[2A_3 \ln(\cosh \omega_0 t) \\
&+ A_4 \tanh^2 \omega_0 t]\tanh \omega_0 t + A_5(1-3\tanh^2 \omega_0 t)\Big\} . \qquad (40)
\end{aligned}
$$

As mentioned in Section **2**, one can substitute Eqs. (35) and (36) into Eq. (31) and then completes the integration of Eq. (38) to obtain Eq. (40). Then,

$$
\begin{aligned}
\dot{x}_1 = a[&2(A_4 - A_3) + (2A_3 - 7A_4)\operatorname{sech}^2 \omega_0 t \\
&+5A_4 \operatorname{sech}^4 \omega_0 t + 2A_3 \ln\left(\cosh \omega_0 t\right)(2 \\
&-3\operatorname{sech}^2 \omega_0 t)]\operatorname{sech}^2 \omega_0 t \\
&-4A_5 a\omega_0 \operatorname{sech}^2 \omega_0 t \tanh \omega_0 t(2-3\tanh^2 \omega_0 t) . \qquad (41)
\end{aligned}
$$

To determine μ_{c1}, one inserts Eq. (29) into Eq.

(24) and yields

$$\int_{-\infty}^{+\infty} \dot{x}_0[\mu_{c1}\dot{x}_0 + x_1\dot{x}_0(\mu_1 - 2\mu_2 x_0) + \dot{x}_1(\mu_{c0} + \mu_1 x_0 - \mu_2 x_0^2 - 2\mu_3\dot{x}_0) - c_2 x_1^2]\,\mathrm{d}\,t = 0\,. \tag{42}$$

By substituting x_0, $\dot{x}_0$, x_1 and $\dot{x}_1$ given in Eqs. (10), (12), (40) and (41) into Eq. (42), the latter can be integrated to be

$$\frac{a^2}{\omega_0}\Big\{B_1 + B_2\,\mathrm{sech}^2\,\omega_0 t + B_3\,\mathrm{sech}^4\,\omega_0 t + B_4\,\mathrm{sech}^6\,\omega_0 t + B_5\,\mathrm{sech}^8\,\omega_0 t + B_6\,\mathrm{sech}^{10}\,\omega_0 t + B_7[\frac{1}{3}\mathrm{sech}^6\,\omega_0 t\ln(\cosh\omega_0 t) - (\frac{1}{15} - \frac{2}{21}\tanh^2\omega_0 t + \frac{1}{27}\tanh^4\omega_0 t)\tanh^2\omega_0 t]\tanh^2\omega_0 t\Big\}\tanh\omega_0 t\Big|_{-\infty}^{+\infty} = \frac{2a^2}{\omega_0}(B_1 - \frac{8}{945}B_7) = 0\,, \tag{43}$$

where

$$B_1 = \frac{16}{21}\mu_{c0}\omega_0^2 A_5 + \frac{8}{15}\mu_{c1}\omega_0^2 + \frac{16}{21}\mu_1 A_5\omega_0^2(\frac{4}{5}a + b) - \frac{16}{21}\mu_2 A_5\omega_0^2(\frac{8}{11}a^2 + \frac{8}{5}ab + b^2) + \frac{128}{315}\mu_3 a\omega_0^2(A_3 + \frac{4}{11}A_4) - \frac{128}{3465}ac_1 A_4 A_5\,, \tag{44}$$

$$B_7 = 16\mu_3 a\omega_0^2 A_3 - 8ac_1 A_3 A_5\,, \tag{45}$$

and the coefficients B_2 to B_6 are listed in Appendix. For nonzero a, Eq. (43) implies

$$B_1 - \frac{8}{945}B_7 = 0\,. \tag{46}$$

By consolidating Eqs.(33),(37),(44) and (45), Eq. (46) yields

$$\mu_{c1} = -\frac{4}{14553}\mu_3 a^2\left\{11\mu_1(6 - \frac{c_1 a}{\omega_0^2}) + 2\mu_2[6(a - 11b) + \frac{c_1 a}{\omega_0^2}(6a + 11b)]\right\}. \tag{47}$$

Finally, the homoclinic solution of Eq. (28) can be obtained as

$$x = a\,\mathrm{sech}^2\,\omega_0 t + b - \varepsilon a\,\mathrm{sech}^2\,\omega_0 t\left\{\frac{\tanh\omega_0 t}{\omega_0}[2A_3\ln(\cosh\omega_0 t) + A_4\tanh^2\omega_0 t] + A_5(3\tanh^2\omega_0 t - 1)\right\} + O(\varepsilon^2)\,, \tag{48}$$

$$\dot{x} = -2a\omega_0\,\mathrm{sech}^2\,\omega_0 t\tanh\omega_0 t + \varepsilon a[2(A_4 - A_3) + (2A_3 - 7A_4)\mathrm{sech}^2\,\omega_0 t + 5A_4\,\mathrm{sech}^4\,\omega_0 t + 2A_3\ln\left(\cosh\omega_0 t\right)(2 - 3\,\mathrm{sech}^2\,\omega_0 t)] - 4\varepsilon A_5 a\omega_0(2 - 3\tanh^2\omega_0 t)\,\mathrm{sech}^2\,\omega_0 t\tanh\omega_0 t + O(\varepsilon^2)\,, \tag{49}$$

$$\mu_c = -(\frac{4}{7}a + b)\mu_1 + (\frac{8}{21}a^2 + b^2 + \frac{8}{7}ab\ )\mu_2 - \varepsilon\frac{4}{14553}\mu_3 a^2\{\ 11\mu_1(6 - \frac{c_1 a}{\omega_0^2}) + 2\mu_2[6(a - 11b) + \frac{c_1 a}{\omega_0^2}(6a + 11b)]\ \} + O(\varepsilon^2)\,. \tag{50}$$

4 Examples

In this section, four examples would be presented to assess the efficacy and accuracy of the present method.

Example 1. Consider the following equation:

$$\ddot{x} + x - 0.4x^2 = \varepsilon(\mu - 0.2x)\dot{x}, \tag{51}$$

which is a case of the oscillator in Eq. (28) with c_1=1, c_2=-0.4, μ_1=-0.2, μ_2=0 and μ_3=0. With ε=1, the example has been studied by both the standard Melnikov method and Belhaq's technique combined with trigonometric solution to second order by multiple scales method [9]. The homoclinic bifurcation curves of the equation in the ω-μ plane were investigated by trigonometric multiple scales method. Here, $\omega = \sqrt{c_1}$ denotes the linear frequency of the linear generating equation.

By using the present method and from Eqs. (12), a = -3.75, b = 2.5 and ω = 0.5. Through Eq. (33), A_3 = 3/28, A_4 = 0 and A_5 = 0. The homoclinic solution of Eq. (51) is then solved to be

$$x = -\frac{15}{4}\operatorname{sech}^2\frac{t}{2} + \frac{5}{2} + \frac{9}{56}\varepsilon \ln(\cosh\frac{t}{2})\operatorname{sech}^2\frac{t}{2}\tanh\frac{t}{2} + O(\varepsilon^2),$$

$$\dot{x} = \frac{15}{4}\operatorname{sech}^2\frac{t}{2}\tanh\frac{t}{2} - \frac{27}{112}\varepsilon \ln(\cosh\frac{t}{2})\operatorname{sech}^2\frac{t}{2}\tanh^2\frac{t}{2} + \frac{9}{112}\varepsilon\left[\ln(\cosh\frac{t}{2}) + \tanh^2\omega t\right]\operatorname{sech}^2\omega t + O(\varepsilon^2).$$

With ε = 1, μ_c = 0.071 428 can be obtained from Eq. (50) or by the standard Melnikov method [9]. The homoclinic orbit is shown in Figure 1. The homoclinic bifurcation curves in the ε-μ_c plane and in the ω-μ plane are plotted in Figure 2 and Figure 3, respectively.

Fig. 1 Homoclinic orbit and limit cycle at μ_c for Eq. (51) with ε=1.0.

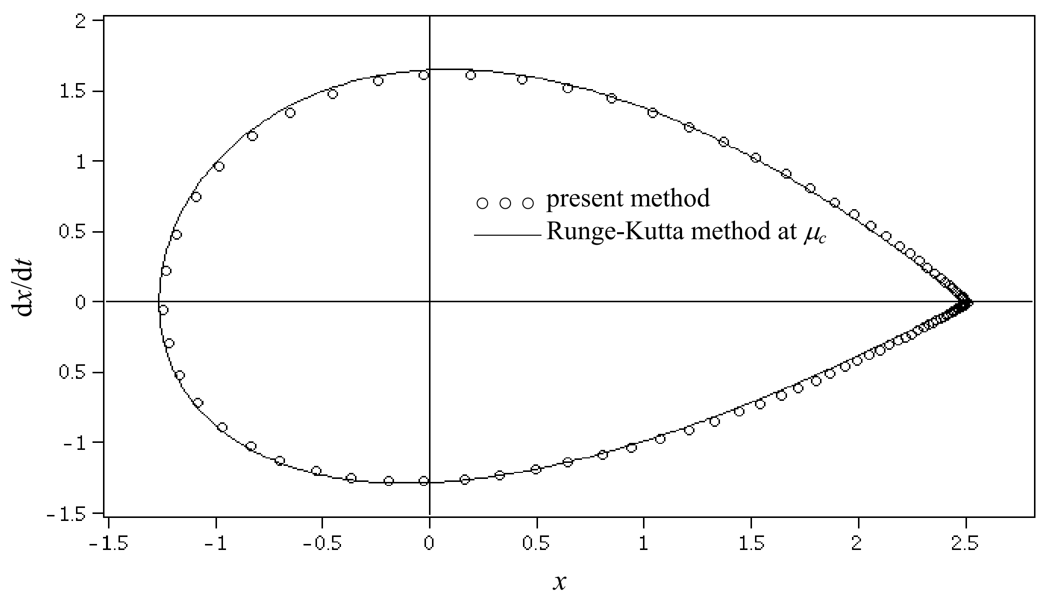


Fig. 2 Homoclinic bifurcation curves in the ε-μ plane for Eq. (51).

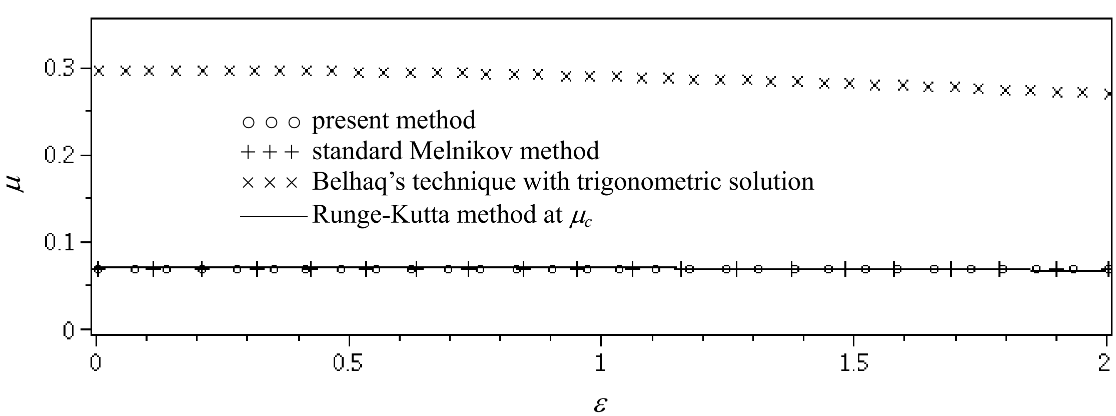
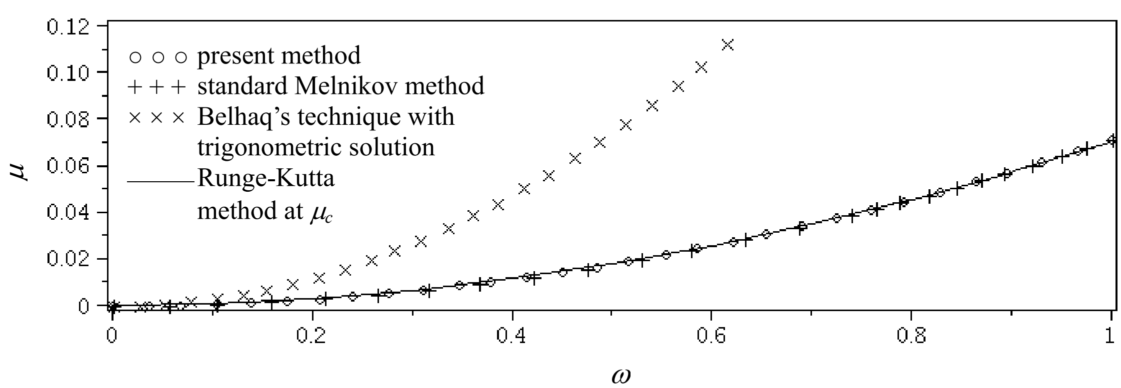


Fig. 3 Homoclinic bifurcation curves in the ω-μ plane for Eq. (51) with ε =1.0.



In this paper, the procedure of using Runge-Kutta integration method to determine the value of μ_c of the homoclinic orbit follows that of Merkin and Needham [14]. Numerical integration is conducted for a given value of ε starting from a value of μ at which there is a limit cycle. It is repeated for increasing or reducing μ until a value of μ is reached such that there is no limit cycle. By successfully reducing the interval of μ within which a limit cycle disappears, a critical value μ_c can be identified such that a limit cycle can be found at $\mu = \mu_c$ but not at $\mu = \mu_c \pm \Delta$ where Δ is a small preset tolerance. Here, Δ is taken to be 10^{-6}. Using this trial-and-error approach, μ_c = 0.070 493. The value is close to but slightly lower than 0.071 428 obtained by the present method.

Example 2. Consider the following equation:

$$\ddot{x} + 0.5x - 2x^2 = \varepsilon(\mu + 2x + 0.2\dot{x})\dot{x}, \tag{52}$$

which is a case of Eq. (28) with $c_1 = 1$ $c_2 = 3$, $\mu_1 = 2.5$, $\mu_2 = 0$ and $\mu_3 = 1$. From Eq. (12), $a = 0.5$, $b = -1/3$ and $\omega = 0.5$. Through Eq. (33), $A_3 = 5/28$, $A_4 = 0$ and $A_5 = -1/12$. The homoclinic solution of Eq. (51) is solved to be

$$x = \frac{1}{2}\operatorname{sech}^2\frac{t}{2} - \frac{1}{3} + \frac{\varepsilon}{2}\left\{\frac{5}{7}\ln(\cosh\frac{t}{2})\tanh\frac{t}{2} - \frac{1}{12}(1 - 3\tanh^2\frac{t}{2})\right\}\operatorname{sech}^2\frac{t}{2} + O(\varepsilon^2),$$

$$\dot{x} = -\frac{1}{2}\operatorname{sech}^2\frac{t}{2}\tanh\frac{t}{2} + \frac{5\varepsilon}{28}[-1 + \operatorname{sech}^2\omega t + (2 - 3\operatorname{sech}^2\frac{t}{2})\ln(\cosh\frac{t}{2})] + \frac{1}{12}(2 - 3\tanh^2\omega t)\operatorname{sech}^2\omega t\tanh\omega t + O(\varepsilon^2).$$

With ε = 0.9, one obtains μ_c = 0.114 560 by Runge-Kutta method, 0.119 047 by standard Melnikov method and 0.112 244 from Eq. (50). The homoclinic orbit is shown in Figure 4. The homoclinic bifurcation curves in the ε-μ plane and in the ω-μ plane are plotted in Figure 5 and Figure 6, respectively. In the figures, the predictions of the standard Melnikov method, Belhaq's technique [9] and Runge-Kutta method are also shown.

Example 3. Consider the following equation:

$$\ddot{x} + 2x - x^2 = \varepsilon(\mu + x - x^2 + 0.1\dot{x})\dot{x}, \tag{53}$$

which is a case of Eq. (28) with $c_1 = 2$, $c_2 = -1$, $\mu_1 = 1$, $\mu_2 = -1$ and $\mu_3 = -0.1$. From Eq. (12), $a = -3$, $b = 2$ and $\omega = 1/\sqrt{2}$. Through Eq. (33), $A_3 = 3/7$, $A_4 = -1$ and $A_5 = -1/20$. The homoclinic solution of Eq. (53) is solved to be

$$x = -3\operatorname{sech}^2\frac{t}{\sqrt{2}} + 2 + 3\varepsilon\operatorname{sech}^2\frac{t}{\sqrt{2}}\left\{[\frac{6}{7}\sqrt{2}\ln(\cosh\frac{t}{\sqrt{2}}) - \tanh^2\frac{t}{\sqrt{2}} - \frac{3}{20}\tanh\frac{t}{\sqrt{2}}]\tanh\frac{t}{\sqrt{2}} + \frac{1}{20}\right\} + O(\varepsilon^2),$$

$$\dot{x} = 3\sqrt{2}\operatorname{sech}^2\frac{t}{\sqrt{2}}\tanh\frac{t}{\sqrt{2}} - 3\varepsilon[-\frac{20}{7} + \frac{55}{7}\operatorname{sech}^2\frac{t}{\sqrt{2}} - 5\operatorname{sech}^4\frac{t}{\sqrt{2}} + \frac{6}{7}(2 - 3\operatorname{sech}^2\frac{t}{\sqrt{2}})\ln(\cosh\frac{t}{\sqrt{2}})] - \frac{3\sqrt{2}}{10}(2 - 3\tanh^2\frac{t}{\sqrt{2}})\operatorname{sech}^2\frac{t}{\sqrt{2}}\tanh\frac{t}{\sqrt{2}} + O(\varepsilon^2).$$

With ε =1.0, one obtains μ_c =0.237 149 by Runge-Kutta method and 0.236 734 from Eq. (50). The homoclinic orbit for is shown in Figure 7 whilst The homoclinic bifurcation curve in the ε-μ plane is plotted in Figure 8. In the figures, comparisons are also made with Runge-Kutta method.

Example 4. Consider the following equation:

$$\ddot{x} - x - 2x^2 = \varepsilon(\mu + x + 3x^2 - 0.5\dot{x})\dot{x}, \tag{54}$$

which is a case of Eq. (28) with c_1=-1 c_2=-2, μ_1=1, μ_2=-3 and μ_3=0.5. From Eqs. (12), a=-3/4, b=0 , ω =1/2. Through Eq. (33), A_3=3/56, A_4=3/16 and A_5=1/16. Then the homoclinic solution of Eq. (54) is solved to be

$$x=-\frac{3}{4}\operatorname{sech}^2\frac{t}{2}-\frac{3}{4}\varepsilon\operatorname{sech}^2\frac{t}{2}\left\{\frac{3}{2}[\frac{1}{7}\ln(\cosh\frac{t}{2})+\frac{1}{4}\tanh^2\frac{t}{2}]\tanh\frac{t}{2}+\frac{1}{16}(3\tanh^2\frac{t}{2}-1)\right\}+O(\varepsilon^2)$$

$$\dot{x}=\frac{3}{4}\operatorname{sech}^2\frac{t}{2}\tanh\frac{t}{2}-\frac{3}{64}\varepsilon[\frac{30}{7}-\frac{135}{7}\operatorname{sech}^2\frac{t}{2}+15\operatorname{sech}^4\frac{t}{2}+6(2-3\operatorname{sech}^2\frac{t}{2})\ln(\cosh\frac{t}{2})]-\frac{3}{32}(2-3\tanh^2\frac{t}{2})\operatorname{sech}^2\frac{t}{2}\tanh\frac{t}{2}+O(\varepsilon^2).$$

With ε =0.9, one obtains μ_c = -0.232 285 by Runge-Kutta method and -0.224 095 from Eq. (50). The homoclinic orbit for the case is shown in Figure 9. The homoclinic bifurcation curve in the ε-μ plane is plotted in Figure 10. Comparisons are also made with Runge-Kutta method.

Fig. 4 Homoclinic orbit and limit cycle at μ_c for Eq.(52) with ε =0.9.

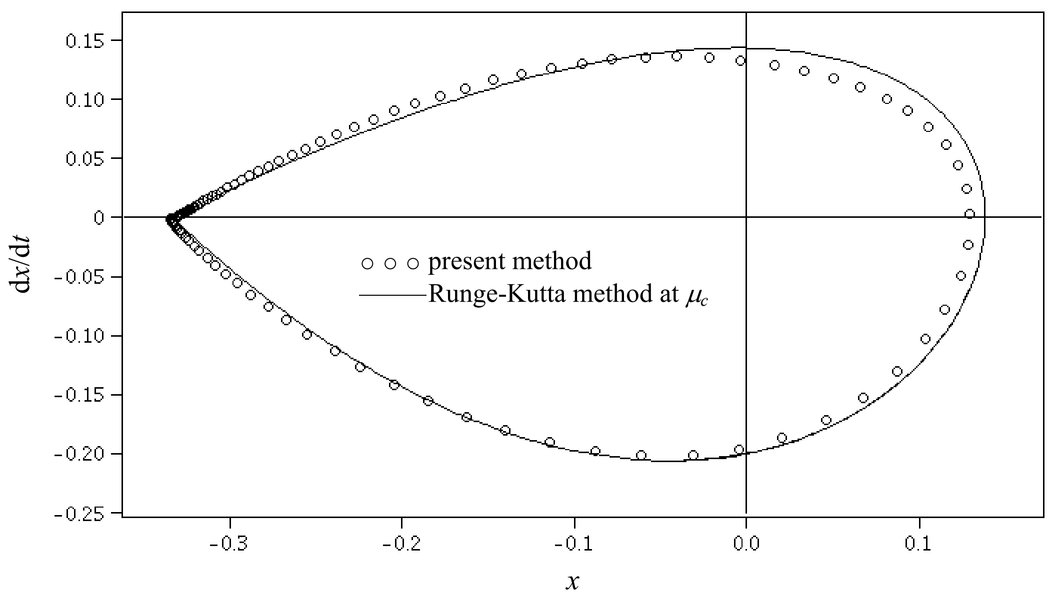


Fig. 5 Homoclinic Bifurcation curves in the ε-μ plane for Eq. (52).

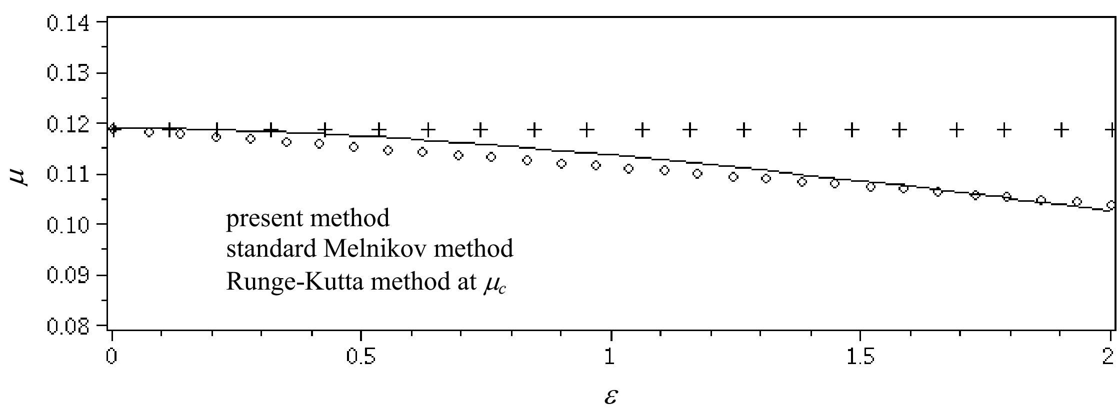


Fig. 6 Homoclinic Bifurcation curves in the ω-μ plane for Eq. (52) with ε =0.9.

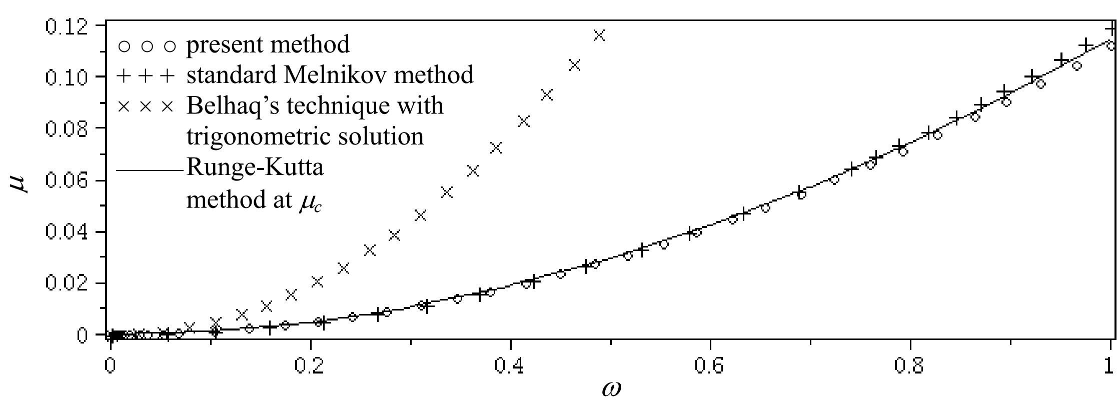


Fig. 7 Homoclinic orbit and limit cycle at μ_c for Eq. (53) with ε =1.0.

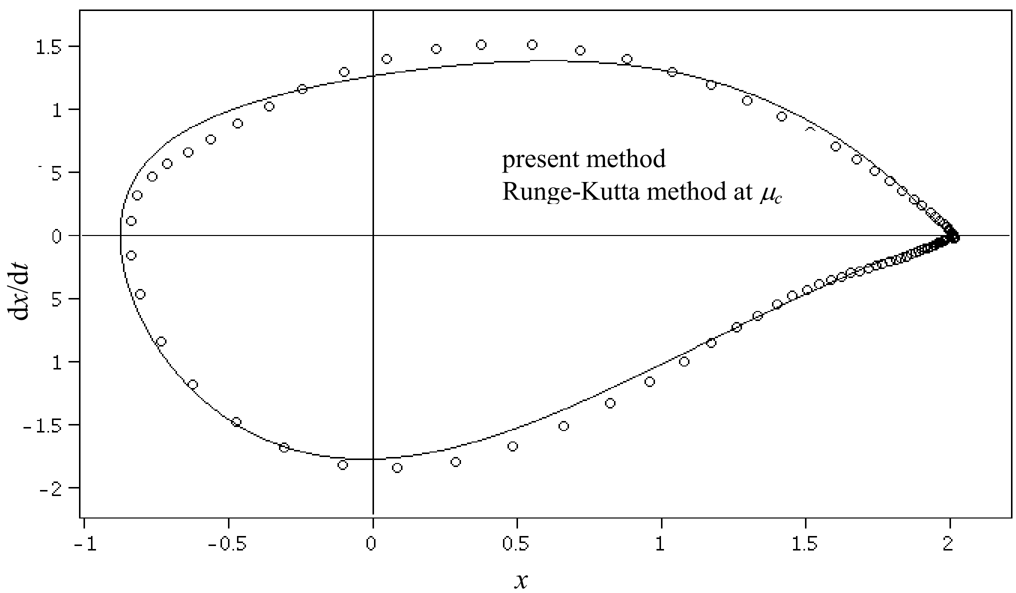


Fig. 8 Homoclinic bifurcation curves in the ε-μ plane for Eq. (53).

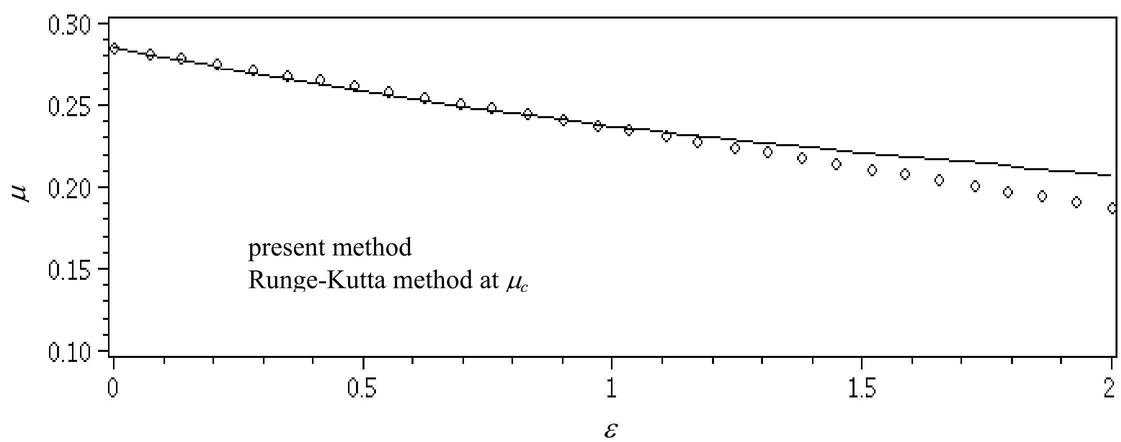


Fig 9. Homoclinic orbit and limit cycle at μ_c for Eq. (54) with ε=0.9.

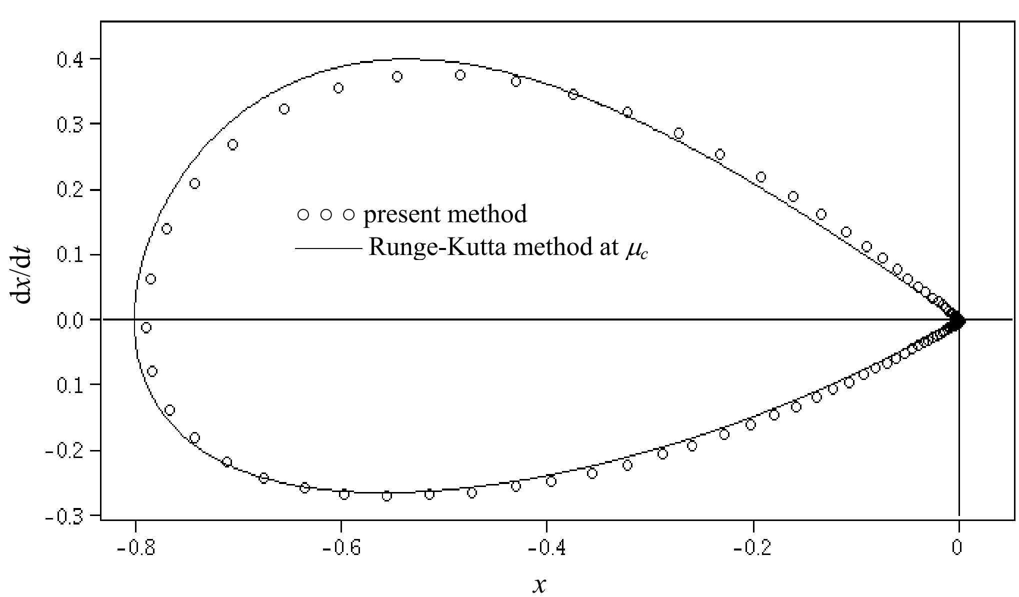
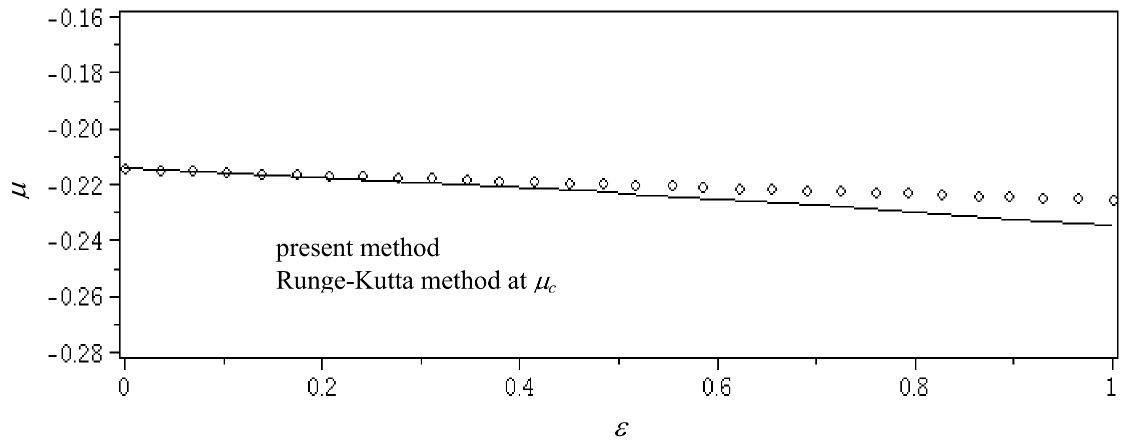


Fig. 10 Homoclinic bifurcation curves in the ε-μ plane for Eq. (54).



It can be seen from Figure 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 that the present predictions for the four examples are in good agreement with those obtained by the Runge-Kutta method even if ε is moderately large. The predicted homoclinic orbits are close to those obtained by Runge-Kutta method at the critical value μ_c. Figures 2, 3, 5 and 6 also indicate that the present method is more accurate in analytically predicting the homoclinic bifurcation than the standard Melnikov method and Belhaq's technique with trigonometric solution. Meanwhile, it is worth noting that the μ_c value predicted by Runge-Kutta method is only a close approximation of the critical value of μ and the result of Runge-Kutta method at μ_c is still a limit cycle which is a periodic solution. At the exact critical value of μ, a homoclinic orbit which is a solution with infinite period should be obtained.

5 Conclusions

(1) The hyperbolic Lindstedt-Poincaré method presented in this paper is an efficient method to construct approximate homoclinic solutions for certain nonlinear autonomous oscillators.
(2) The critical value of μ, under which there exists a homoclinic solution of the nonlinear autonomous oscillator, can be approximately determined by the present perturbation procedure.
(3) Typical examples show that good accuracy of the homoclinic orbits obtained by present method even for moderately large value of ε.
(4) The present hyperbolic Lindstedt-Poincaré method can be generalized to determine heteroclinic solutions for certain strongly nonlinear autonomous oscillators.

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Appendix

The coefficients B_2, B_3, B_4, B_5 and B_6 in Eq. (43) are:

$$B_2 = \frac{8}{21}\mu_{c0}\omega_0^2 A_5 + \frac{4}{15}\mu_{c1}\omega_0^2 + \frac{8}{21}\mu_1 A_5\omega_0^2(\frac{4}{5}a + b)$$
$$-\frac{8}{21}\mu_2 A_5\omega_0^2(\frac{104}{165}a^2 + \frac{8}{5}ab + b^2)$$
$$+\frac{64}{315}\mu_3 a\omega_0^2(A_3 + \frac{4}{11}A_4) - \frac{64}{3465}ac_1 A_4 A_5,$$

$$B_3 = \frac{16}{7}\mu_{c0}\omega_0^2 A_5 - \frac{4}{5}\mu_{c1}\omega_0^2 + \frac{8}{7}\mu_1 A_5\omega_0^2(\frac{1}{5}a + 2b)$$
$$-\frac{16}{7}\mu_2 A_5\omega_0^2(\frac{13}{165}a^2 + \frac{1}{5}ab + b^2)$$
$$+\frac{16}{105}\mu_3 a\omega_0^2(A_3 + \frac{4}{11}A_4) - \frac{16}{1155}ac_1 A_4 A_5,$$

$$B_4 = -\frac{24}{7}\mu_{c0}\omega_0^2 A_5 + \frac{4}{7}\mu_1 A_5\omega_0^2(5a - 6b)$$
$$-\frac{8}{7}\mu_2 A_5\omega_0^2(\frac{13}{99}a^2 + \frac{8}{3}ab - 3b^2)$$
$$-\frac{40}{63}\mu_3 a\omega_0^2(4A_3 - \frac{47}{11}A_4) + \frac{916}{693}ac_1 A_4 A_5,$$

$$B_5 = -4\mu_1 A_5 a\omega_0^2 - \frac{16}{3}\mu_2 A_5\omega_0^2(\frac{7}{33}a^2 - ab)$$
$$+\frac{16}{9}\mu_3 a\omega_0^2(A_3 - \frac{41}{11}A_4) - \frac{232}{99}ac_1 A_4 A_5,$$

$$B_6 = \frac{24}{11}\mu_2 A_5 a^2\omega_0^2 + \frac{40}{11}\mu_3 A_4 a\omega_0^2.$$